INTERPOLATORY FUNCTION THEORY

BY

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PREFACE

The title of this work is not, perhaps, a very happy one, but I cannot think of a better. The intention is to gather together a number of results on interpolation, finite differences and successive derivatives which seem to have something in common, avoiding as far as possible all matter which has appeared in other books and all matter concerned with formulae in themselves rather than with their applications. The field is large, and the length of a tract is strictly limited. Much interesting work of Pólya, Valiron, Carmichael, Bochner, Nörlund, Ferrar, de la Vallée Poussin, R. Lagrange and many others has been omitted entirely as there was no room to do justice to it. Nor has anything been said about the important branch of function theory described in Walsh's recent tract *Approximation by polynomials in the complex domain* (Paris, 1935). This, though intimately connected with interpolation, belongs to a different order of ideas.

Some of the material has not been published before, in particular the definition of the order and type of a set of polynomials (which supersedes the definition of order given in the writer's paper **11**), substantial parts of Theorems 9, 15, 17, 19 and 20, and the proofs of Theorems 8 and 13.

Since the tract was written an important paper by A. J. MacIntyre has appeared*, in which the main results of Chapter V are proved and extended to meromorphic functions by Picard-Schottky methods.

My thanks are due to Professor E. T. Copson for reading the proofs and to the staff of the University Press for their excellent work. Finally I wish to record my gratitude to my father, Professor E. T. Whittaker, more especially as his memoir on the cardinal function was the starting point of my interest in these matters.

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Liverpool
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* "A theorem concerning meromorphic functions of finite order", *Proc. London Math. Soc.* 39 (1935), 282–294.

INTRODUCTION

§ 1. Preliminary results.

Most of the theorems proved in this work concern integral and meromorphic functions and a rudimentary knowledge of their properties will be assumed. For integral functions we need only the results leading up to the expression of a function of finite order as a canonical product and the simpler relations connecting the maximum modulus with the coefficients and maximum term of the Taylor series, in particular the equations

$$(1\cdot 1) \qquad \rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \overline{\lim_{n\to\infty}} \frac{n\log n}{\log |a_n|^{-1}},$$

$$(1\cdot 2) \quad \sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\rho}} = \frac{1}{e\rho}\left(\overline{\lim_{n\to\infty}}\, n^{1/\rho}\,|a_n|^{1/n}\right)^{\rho} \qquad (0<\rho<\infty)$$

for the order ρ and the type σ. All this will be found in the first sixty pages of Valiron's book (**1**).

For meromorphic functions it is sufficient to know the first thirty pages of Nevanlinna (**1**)—the definition and fundamental properties of $T(r)$—and one other result (**14**),* which in some ways takes the place of (1·1). This is as follows.

Let κ denote the exponent of convergence of the poles p_n of a meromorphic function $f(z)$, i.e. the lower bound of numbers k such that $\Sigma\,|\,p_n\,|^{-k}$ converges. We may suppose that κ is finite. Take a number $h>\kappa$ and surround each pole p_n with a circle of radius $|\,p_n\,|^{-h}$. These circles will now group themselves into "nebulae", each circle of a nebula having at least one point in common with some other circle of the nebula, but no point in common with other nebulae. Corresponding to each nebula there will be a "pole-cluster" and it can be shown that each pole-cluster contains only a finite number of poles. The grouping into pole-clusters may depend on the particular value of h chosen.

* Numbers without names attached refer to the writer's papers.

Now group together the terms in the Mittag-Leffler expansion of $f(z)$, which correspond to the different pole-clusters. We obtain a series

$$(1\cdot 3)\quad f(z)=g(z)+\Sigma\left\{\frac{P(z)}{(z-b_0)^{\lambda_0}(z-b_1)^{\lambda_1}\dots(z-b_k)^{\lambda_k}}+Q(z)\right\},$$

where $g(z)$ is an integral function, the summation is over the pole-clusters, and $P(z)$, $Q(z)$ are polynomials. Write

$$(1\cdot 4)\quad P(z)=P_0+P_1z+\dots+P_{\mu-1}z^{\mu-1}\qquad(\mu=\lambda_0+\lambda_1+\dots+\lambda_k),$$

$$(1\cdot 5)\quad P=\max_{0\leq i<\mu}|P_i|,\quad \delta(r)=\max_{|z|=r}|P(z)|\qquad(|b_0|=r).$$

$$(1\cdot 6)\qquad \tau=\overline{\lim_{r\to\infty}}\frac{\log^+\log^+P}{\log r},\quad \tau_1=\overline{\lim_{r\to\infty}}\frac{\log^+\log^+\delta(r)}{\log r}.$$

$Q(z)$ is the sum of the first n terms of the Taylor series of $-P(z)(z-b_0)^{-\lambda_0}\dots(z-b_k)^{-\lambda_k}$, n being chosen so that

$$(1\cdot 7)\qquad \log^+P+Ap(r)\leq n\leq\log^+P+Bp(r),$$

where $p(r)$ denotes the number of poles in $|z|\leq r$, and A, B are constants.

If these conditions are satisfied (1·3) is called an expression of $f(z)$ in *normal form* and the result in question is as follows:

Theorem A. *If $f(z)$, a meromorphic function of order ρ, is expressed in normal form* (1·3), *then*

$$(1\cdot 8)\qquad \rho=\max(\sigma,\tau,\kappa)=\max(\sigma,\tau_1,\kappa),$$

where σ is the order of $g(z)$ and κ, τ, τ_1 are defined above.

The order of a part of the series (1·3) *cannot exceed ρ.*

The theorem affords a means of dealing with difficulties which arise when some of the poles of a meromorphic function are very close together. Similar difficulties are encountered in interpolation. If $A_1, A_2, \dots$ is any set of complex numbers and $e_1, e_2, \dots$ any set such that $|e_n|\to\infty$, it is possible to find an integral function $f(z)$ such that

$$(1\cdot 9)\qquad f(e_n)=A_n\qquad(n=1,2,\dots).$$

For there exists an integral function $\phi(z)$ with simple zeros at the e's, and we may take

$$(1\cdot10) \qquad f(z)=\phi(z)\sum_{n=1}^{\infty}\frac{A_n z^{s_n}}{\phi'(e_n)\,e_n^{s_n}(z-e_n)},$$

the integers $s_1, s_2, \ldots$ being chosen so that the series converges uniformly in any finite region of the plane. The difficulties arise when we try to determine the order of $f(z)$. If some of the e's are very close together the order may be high, even if the A's are comparatively small and the e's tend to infinity rapidly. The question has been investigated by Borel (**1**), Pólya (**1**), Mursi and Winn (**1, 2**), and MacIntyre and Wilson (**1**).

CHAPTER I

SERIES OF POLYNOMIALS

§ 2. Basic sets of polynomials.

Let $p_0(z), p_1(z), \ldots$ be a set of polynomials. An expression of the form

$$A_0 p_0(z) + A_1 p_1(z) + \ldots + A_k p_k(z)$$

is called a finite linear combination of the polynomials, and the latter are said to be linearly independent if no such combination is identically zero unless all the constants $A_0, \ldots, A_k$ are zero. If, in addition, every polynomial can be expressed as a finite linear combination of the given set, we shall say that the polynomials form a *basic set*. Thus the definition is as follows:

Definition. *Polynomials $p_0(z), p_1(z), \ldots$ form a basic set if every polynomial can be expressed in one and only one way as a finite linear combination of them.*

A few of the simpler properties of basic sets follow from the definition without difficulty. For example, if $p_0(z), p_1(z), \ldots$ is a basic set, and $c_0, c_1, \ldots$ are any constants, then

$$1, \quad \int_{c_0}^{z} p_0(t)\,dt, \quad \int_{c_1}^{z} p_1(t)\,dt, \ldots$$

is a basic set; moreover $p_0'(z), p_1'(z), \ldots$ form a basic set together with one additional polynomial (which is identically zero if one of the given set is a constant).

In order to proceed further it is necessary to recall some properties of infinite matrices. Let $\mathbf{A}$ be a matrix of the form

$$(2\cdot1) \qquad \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdot \\ a_{10} & a_{11} & a_{12} & \cdot \\ a_{20} & a_{21} & a_{22} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

$\mathbf{A}$ is said to be *row-finite* if each row contains only a finite number of non-zero elements. A matrix $\mathbf{B}$ which satisfies the equations

$$\mathbf{AB} = \mathbf{I}, \quad \mathbf{BA} = \mathbf{I},$$

where $\mathbf{I}$ is the unit matrix,* is called a *reciprocal* of $\mathbf{A}$. If it only satisfies the first (second) equation it is called a *right-hand* (*left-hand*) *reciprocal*. Multiplication of matrices of the form (2·1) is not in general associative. For example, if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are respectively

$$\begin{bmatrix} 1 & -1 & 1 & . \\ 0 & 0 & 0 & . \\ 0 & 0 & 0 & . \\ . & . & . & . \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & . \\ 1 & 1 & 0 & . \\ 0 & 1 & 1 & . \\ . & . & . & . \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & . \\ -1 & 1 & 0 & . \\ 1 & -1 & 1 & . \\ . & . & . & . \end{bmatrix},$$

it will be found that

$$\mathbf{A}(\mathbf{BC})=\mathbf{A}, \quad (\mathbf{AB})\mathbf{C}=\mathbf{0}.$$

It is known,† however, that the multiplication of row-finite matrices is associative and that in any associative algebra an element a has a unique reciprocal if it has both a right-hand reciprocal and a left-hand reciprocal; and, moreover, that a has a unique two-sided reciprocal in that algebra if it has a unique reciprocal on one side.

Write

(2·2) $$p_i(z)=p_{i0}+p_{i1}z+p_{i2}z^2+\dots,$$

(2·3) $$\mathbf{P}=\begin{bmatrix} p_{00} & p_{01} & p_{02} & . \\ p_{10} & p_{11} & p_{12} & . \\ p_{20} & p_{21} & p_{22} & . \\ . & . & . & . \end{bmatrix},$$

so that $\mathbf{P}$ is row-finite. The condition for a basic set is then as follows (**11**):

Theorem 1. *In order that $p_0(z)$, $p_1(z)$, ... should form a basic set it is necessary and sufficient that $\mathbf{P}$ should have a row-finite reciprocal.*

In view of the preceding remarks the condition is equivalent to the statement that there is a unique row-finite matrix $\mathbf{\Pi}$ satisfying the equation

(2·4) $$\mathbf{\Pi P}=\mathbf{I}.$$

* I.e. that in which $a_{ii}=1$, $a_{ij}=0$ $(i \neq j)$.

† Dienes (**1**), 416.

To show that this condition is necessary, write down the equations expressing 1, z, z^2, ... as linear combinations of $p_0(z)$, $p_1(z)$, ...; say

$$1 = \pi_{00} p_0(z) + \pi_{01} p_1(z) + \dots, \tag{2·5}$$

$$z = \pi_{10} p_0(z) + \pi_{11} p_1(z) + \dots, \tag{2·6}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Each of these series is finite, so the matrix $\mathbf{\Pi} = \{\pi_{ij}\}$ is row-finite. In full (2·5) is

$$1 = \pi_{00}(p_{00} + p_{01} z + \dots) + \pi_{01}(p_{10} + p_{11} z + \dots) + \dots,$$

so that, on equating coefficients,

$$1 = \pi_{00} p_{00} + \pi_{01} p_{10} + \pi_{02} p_{20} + \dots,$$

$$0 = \pi_{00} p_{01} + \pi_{01} p_{11} + \pi_{02} p_{21} + \dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

These, together with other equations derived in the same way, are equivalent to

$$\mathbf{\Pi P} = \mathbf{I}.$$

Again there cannot be a row-finite matrix $\mathbf{R}$, other than $\mathbf{\Pi}$, such that

$$\mathbf{RP} = \mathbf{I}.$$

For, if so, let $\mathbf{\Pi} - \mathbf{R} = \{\alpha_{ij}\}$. As not all elements of this matrix are zero, let there be a non-zero element in, say, the first row of $\mathbf{\Pi} - \mathbf{R}$. Let α_{0k} be the last non-zero element in this row. Now

$$(\mathbf{\Pi} - \mathbf{R})\mathbf{P} = \mathbf{0}$$

and so

$$\alpha_{00} p_{00} + \alpha_{01} p_{10} + \dots + \alpha_{0k} p_{k0} = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\alpha_{00} p_{0l} + \alpha_{01} p_{1l} + \dots + \alpha_{0k} p_{kl} = 0,$$

the final equation being the last in which the p's are not all zero. Such an equation exists since $\mathbf{P}$ is row-finite. On multiplying these equations by 1, z, ..., z^l and adding, we get

$$\alpha_{00} p_0(z) + \alpha_{01} p_1(z) + \dots + \alpha_{0k} p_k(z) \equiv 0,$$

and this is impossible, since $\alpha_{0k} \neq 0$, whereas $p_0(z)$, $p_1(z)$, ... are linearly independent. This completes the proof of the necessity of the condition.

To prove its sufficiency, assume the truth of (2·4) and write

$$\Pi_i = \pi_{0i} + \pi_{1i}\frac{d}{dz} + \frac{\pi_{2i}}{2!}\frac{d^2}{dz^2} + \dots \qquad (i = 0, 1, 2, \dots).$$

, is easy to verify that, if $f(z)$ is any polynomial,

$$f(z) = p_0(z)\,\Pi_0 f(0) + p_1(z)\,\Pi_1 f(0) + \dots,$$

1e series being finite since $\mathbf{\Pi}$ is row-finite. In general each Π_i is 1 infinite operator, but the question of convergence does not)me in as yet, since, if $f(z)$ is a polynomial, the series

$$\Pi_i f(0) = (\Pi_i f(z))_{z=0} = \pi_{0i} f(0) + \pi_{1i} f'(0) + \dots$$

finite.

Thus every polynomial can be expressed in at least one way as finite linear combination of $p_0(z)$, $p_1(z)$, ... Now if there is a)lynomial $g(z)$ which can be so expressed in more than one way, 1is must be true of at least one of the polynomials $1, z, z^2, \dots$. or, if $g(z)$ is of degree N and $1, z, \dots, z^{N-1}$ are all only expressible ı one way, then z^N must be expressible in more than one way. hus there would be more than one set of equations like (2·5), !·6). Each such set would lead to a row-finite matrix $\mathbf{\Pi}$ satis-'ing (2·4), and this is contrary to hypothesis.

Basic sets can be classified by means of a function $U(n)$ efined as

(2·7) *the number of polynomials* $p_0(z)$, $p_1(z)$, ... *of degree less an* n.

The condition that the representation of an arbitrary poly-omial is to be unique shows that the members of a basic set must e linearly independent in the finite sense, and this clearly aplies that

(2·8) $$U(n) \leqslant n \qquad (n \geqslant 1).$$

The most common case is

(2·9) $$U(n) = n \qquad (n \geqslant 1),$$

e. $p_n(z)$ is of degree n. A set of polynomials satisfying (2·9) will e called a *simple* set. Such a set is necessarily a basic set. All the rdinary kinds of polynomials, e.g. those of Legendre, Laguerre, [ermite, Bernoulli, form simple sets.

Failing (2·9), it may be that

(2·10) $U(n_r) = n_r$, for an infinite sequence $n_1, n_2, \ldots$.

A basic set with this property will be called a *regular* set. It is easy to see that any set of linearly independent polynomials satisfying (2·10) is regular.

In the case of a general basic set nothing can be said about $U(n)$ except that it satisfies (2·8) and is an increasing function tending to infinity. For it can be shown that $U(n)$ can increase arbitrarily slowly.

Basic sets of operators. Again, let

$$(2\cdot 11)\qquad \Pi_i = \pi_{0i} + \pi_{1i}\frac{d}{dz} + \frac{\pi_{2i}}{2!}\frac{d^2}{dz^2} + \cdots \qquad (i = 0, 1, 2, \ldots)$$

be any set of operators and let

$$(2\cdot 12)\qquad \mathbf{\Pi} = \begin{bmatrix} \pi_{00} & \pi_{01} & \pi_{02} & \cdot \\ \pi_{10} & \pi_{11} & \pi_{12} & \cdot \\ \pi_{20} & \pi_{21} & \pi_{22} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

be called their matrix. (Note that the coefficients π_{ji} associated with a particular Π_i are in the same *column* of the matrix, whereas the coefficients p_{ij} associated with a particular polynomial $p_i(z)$ are in the same *row* of $\mathbf{P}$.) It is often convenient to speak of $\Pi_0 f(0)$, $\Pi_1 f(0)$, ... as operators; e.g. the matrix of the operators

$$f(0),\quad f'(1),\quad f''(2),\quad \ldots$$

is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 2 & 2 & 0 & \cdot \\ 0 & 3 & 12 & 6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

since

$$f'(1) = f'(0) + \frac{2}{2!}f''(0) + \frac{3}{3!}f'''(0) + \ldots, \text{ etc.}$$

Π_0, Π_1, ... will be said to form a *basic set of operators* if they are associated in the manner described above with a basic set of polynomials. If the latter set is simple, or regular, the set of operators will be said to be simple, or regular.

The condition that a set of operators be basic follows from Theorem 1.

$\mathbf{L}_{101}$. *In order that* $\Pi_0, \Pi_1, \ldots$ *should form a basic set of operators, it is necessary and sufficient that their matrix* $\mathbf{\Pi}$ *should be row-finite and have a row-finite reciprocal.*

The series

$$(2\cdot 13) \qquad p_0(z)\,\Pi_0 f(0) + p_1(z)\,\Pi_1 f(0) + \ldots$$

will be called the associated *basic series*. It has been seen that it represents all polynomials.

Regular sets of polynomials and operators. A basic set of polynomials has been defined to be regular if

$$U(n_r) = n_r, \text{ for an infinite sequence } n_1, n_2, \ldots.$$

It is easy to see that, if n_r is one of these integers, the matrix of the polynomials is of the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_r & \mathbf{0} \\ \mathbf{X} & \mathbf{Y} \end{bmatrix},$$

where $\mathbf{P}_r$ is a square matrix of order n_r, and that the matrix of the operators is of the form

$$\mathbf{\Pi} = \begin{bmatrix} \mathbf{P}_r^{-1} & \mathbf{0} \\ \mathbf{Z} & \mathbf{T} \end{bmatrix}.$$

Thus, in the case of a regular set of polynomials, it is possible to calculate the first n_r^2 elements of $\mathbf{\Pi}$ by forming the reciprocal of $\mathbf{P}_r$, the matrix consisting of the first n_r^2 elements of $\mathbf{P}$. The same is of course true for a regular set of operators. As an example, let

$$(2\cdot 14) \qquad \Pi_{2n} f(0) = f^{(2n)}(1), \quad \Pi_{2n+1} f(0) = f^{(2n)}(0) \qquad (n \geq 0),$$

so that
$$\mathbf{\Pi} = \begin{bmatrix} 1 & 1 & 0 & 0 & . \\ 1 & 0 & 0 & 0 & . \\ 1 & 0 & 2 & 2 & . \\ 1 & 0 & 6 & 0 & . \\ . & . & . & . & . \end{bmatrix}.$$

The reciprocal can be calculated by successively truncating $\mathbf{\Pi}$ into matrices of 2, 4, 6, ... rows and columns. We find that

$$\mathbf{P}=\begin{bmatrix} 0 & 1 & 0 & 0 & . \\ 1 & -1 & 0 & 0 & . \\ 0 & -\frac{1}{6} & 0 & \frac{1}{6} & . \\ 0 & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{6} & . \\ . & . & . & . & . \end{bmatrix},$$

so that the basic polynomials are

$$z, \quad 1-z, \quad \tfrac{1}{6}(-z+z^3), \quad \tfrac{1}{6}(-2z+3z^2-z^3), \quad \ldots.$$

The resulting series was introduced by Lidstone (**1**), who showed that all polynomials could be represented by it.

§ 3. The convergence of basic series.

The convergence properties of a basic series depend to a large extent on the particular polynomials concerned. In some cases a comparatively simple expression can be found for the remainder term. The Gregory-Newton series, discussed in Chapter IV, is an example. It is always possible to find sufficient conditions for convergence by rearranging the Taylor series of $f(z)$. Write

$$(3\cdot1) \qquad M_i(R)=\max_{|z|=R}|p_i(z)|.$$

Now, formally,

$$\begin{aligned} f(z) &= \sum_0^\infty a_n z^n \\ &= \sum_0^\infty a_n \{\pi_{n0}p_0(z)+\pi_{n1}p_1(z)+\ldots\} \\ &= p_0(z)\sum_0^\infty a_n\pi_{n0}+p_1(z)\sum_0^\infty a_n\pi_{n1}+\ldots \\ &= p_0(z)\,\Pi_0 f(0)+p_1(z)\,\Pi_1 f(0)+\ldots. \end{aligned}$$

If $|z|\leqslant R$ these operations will be legitimate if

$$\sum_{n=0}^\infty |a_n|\{\sum_i |\pi_{ni}|M_i(R)\}$$

is convergent. Write

$$(3\cdot2) \qquad \omega_n(R)=\sum_i |\pi_{ni}|M_i(R).$$

Then the following result has been established:

L$_{102}$. *If $f(z) = \Sigma a_n z^n$ is such that*

(3·3) $$\sum_{n=0}^{\infty} |a_n| \, \omega_n(R)$$

converges, the basic series

(3·4) $$p_0(z) \Pi_0 f(0) + p_1(z) \Pi_1 f(0) + \dots$$

converges absolutely and uniformly to $f(z)$ in $|z| \leqslant R$.

It follows that a basic series represents all integral functions of sufficiently slow growth. More precisely:

L$_{103}$. *Associated with a basic series there is a function $\phi(n)$ with the following properties:*

(i) $\phi(n) > 0 \qquad (n \geqslant 0)$;

(ii) *if $f(z) = \Sigma a_n z^n$ satisfies the condition*

(3·5) $$|a_n| < \phi(n) \qquad (n \geqslant n_0),$$

the basic series converges to $f(z)$ absolutely and uniformly in any finite region of the plane.

It is sufficient to take

(3·6) $$\phi(n) = \frac{1}{n^2 \omega_n(n)}.$$

To obtain the best quantitative results a more subtle choice of $\phi(n)$ is called for. Let us first define the *order* and *type* of a basic set.

Definition. *The order of a basic set of polynomials or operators is*

(3·7) $$\omega = \lim_{R \to \infty} \omega(R),$$

where

(3·8) $$\omega(R) = \overline{\lim_{n \to \infty}} \frac{\log \omega_n(R)}{n \log n}.$$

If $0 < \omega < \infty$, the type is

(3·9) $$\gamma = \lim_{R \to \infty} \gamma(R),$$

where

(3·10) $$\left\{ \frac{\omega \gamma(R)}{e} \right\}^{\omega} = \overline{\lim_{n \to \infty}} \{\omega_n(R)\}^{1/n} n^{-\omega}.$$

It is evident that $\omega(R)$, $\gamma(R)$ are increasing functions, so that the limits (3·7), (3·9) exist. A set of order ω', type γ' will be said to be *of smaller increase* than one of order ω, type γ, if $\omega' < \omega$ or if $\omega' = \omega$ but $\gamma' < \gamma$.

L$_{104}$. *If* $\{p_n(z)\}$ *is a basic set of order* ω, *type* γ, *and* $A\ (\neq 0)$, B *are any constants*, $\{p_n(Az+B)\}$ *is a basic set of order* ω, *type* $|A|^{-1/\omega}\gamma$.

Write
$$p_n(Az+B) = p_n^\star(z),$$
and let $p_{ij}^\star$, $M_n^\star(R)$, $\omega_n^\star(R)$, etc., refer to $\{p_n^\star(z)\}$, so that
$$\sum_k p_{nk}^\star z^k = \sum_l p_{nl}(Az+B)^l,$$
and
$$p_{nk}^\star = \sum_{l\ k}^{\infty} p_{nl} A^k B^{l-k} \binom{l}{k},$$
i.e.
$$\mathbf{P}^\star = \mathbf{PA},$$
where
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & . \\ B & A & 0 & . \\ B^2 & 2BA & A^2 & . \\ . & . & . & . \end{bmatrix}.$$
Hence
$$\mathbf{\Pi}^\star = (\mathbf{P}^\star)^{-1} = \mathbf{A}^{-1}\mathbf{P}^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 & . \\ -\dfrac{B}{A} & \dfrac{1}{A} & 0 & . \\ \dfrac{B^2}{A^2} & -\dfrac{2B}{A^2} & \dfrac{1}{A^2} & . \\ . & . & . & . \end{bmatrix} \mathbf{\Pi},$$
and
$$\pi_{nk}^\star = \sum_{l=0}^{n} \binom{n}{l} (-B)^{n-l} A^{-n} \pi_{lk}.$$
Thus
$$|\pi_{nk}^\star| \leqslant \left|\frac{B}{A}\right|^n \sum_{l=0}^{n} \binom{n}{l} |B|^{-l} |\pi_{lk}|.$$
Moreover
$$M_k^\star(R) \leqslant M_k(2|A|R) \qquad (R \geqslant R_0),$$
so that, if $R \geqslant R_0$,

$$(3{\cdot}11)\quad \omega_n^\star(R) \leqslant \left|\frac{B}{A}\right|^n \sum_k M_k(2|A|R) \sum_{l=0}^{n} \binom{n}{l} |B|^{-l} |\pi_{lk}|$$
$$= \left|\frac{B}{A}\right|^n \sum_{l=0}^{n} \binom{n}{l} |B|^{-l} \omega_l(2|A|R).$$

Now, if $\alpha > \omega$,

$$\omega_n(2\,|\,A\,|\,R) < Kn^{\alpha n} \qquad (n \geqslant 0),$$

where K is independent of n, so that

$$\omega_n^\star(R) < K\left|\frac{B}{A}\right|^n \sum_{l=0}^{n}\binom{n}{l}|\,B\,|^{-l}n^{\alpha n} = K\left(\frac{1+|\,B\,|}{|\,A\,|}\right)^n n^{\alpha n},$$

whence $$\omega^\star(R) \leqslant \alpha,$$

and so $$\omega^\star \leqslant \omega.$$

On the other hand,

$$p_n^\star\left(\frac{z}{A}-\frac{B}{A}\right) = p_n(z), \tag{3·12}$$

so that $$\omega \leqslant \omega^\star.$$

Again, if $$\beta > \{\omega\gamma(R)\,e^{-1}\}^{\omega},$$

then $$\omega_n(2\,|\,A\,|\,R) < Kn^{n\omega}\beta^n \qquad (n \geqslant 0),$$

so that, in (3·11),

$$\omega_n^\star(R) < K\left|\frac{B}{A}\right|^n \sum_{l=0}^{n}\binom{n}{l} l^{l\omega}\beta^l\,|\,B\,|^{-l}$$

$$< K\left|\frac{B}{A}\right|^n \sum_{l=0}^{n}\binom{n}{l} n^{l\omega}\beta^l\,|\,B\,|^{-l} = K\,|\,A\,|^{-n}\, n^{n\omega}(|\,B\,|\,n^{-\omega}+\beta)^n,$$

whence $$\{\omega\gamma^\star(R)\,e^{-1}\}^{\omega} \leqslant \beta\,|\,A\,|^{-1},$$

and so $$\gamma^\star(R) \leqslant \omega^{-1}e\beta^{1/\omega}\,|\,A\,|^{-1/\omega},$$

and $$\gamma^\star \leqslant \gamma\,|\,A\,|^{-1/\omega}.$$

In the same way (3·12) gives

$$\gamma \leqslant \gamma^\star\,|\,A\,|^{1/\omega}.$$

L_{105}. *If $\{p_n(z)\}$ is a basic set of order ω, type γ, and $f(z)$ is an integral function of increase less than order $1/\omega$, type $1/\gamma$, the basic series converges absolutely and uniformly to $f(z)$ in any finite region of the plane.*

The limitation on $f(z)$ means that its order ρ and its type σ satisfy either

$$\text{(i)}\ \rho < \frac{1}{\omega} \quad \text{or} \quad \text{(ii)}\ \rho = \frac{1}{\omega},\ \ \sigma < \frac{1}{\gamma}.$$

In case (ii) its Taylor coefficients satisfy the inequality

$$|\,a_n\,| < (\sigma_1 e\omega^{-1}n^{-1})^{n\omega} \qquad (n \geqslant n_1),$$

where $\sigma_1 > \sigma$. Again, if $\gamma_1 > \gamma$,

$$\omega_n(R) < (\gamma_1 \omega e^{-1} n)^{n\omega} \qquad (n \geqslant n_2),$$

and so
$$|a_n|\, \omega_n(R) < (\sigma_1 \gamma_1)^{n\omega} \qquad (n \geqslant n_3).$$

σ_1, γ_1 can be chosen so that $\sigma_1\gamma_1 < 1$, since $\sigma\gamma < 1$, and the result now follows from L_{102}. Case (i) is easily dealt with.

The order of a set of polynomials is naturally connected with the size of their coefficients. The result which follows illustrates the method by which the relationship can be investigated.

Simple sets of polynomials with bounded coefficients. In the case of a simple set of polynomials, i.e. when $p_n(z)$ is of degree n, there is no loss of generality in taking the coefficient of z^n in $p_n(z)$ to be unity. It will be assumed that this is the case. This implies that Π_n is of the form

$$\frac{1}{n!}\frac{d^n}{dz^n} + \frac{\pi_{n+1,n}}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}} + \dots \tag{3·13}$$

and vice versa.

If the remaining coefficients of the polynomials satisfy an inequality

$$|p_{ij}| \leqslant M, \tag{3·14}$$

the basic series has simple properties.

We need a lemma on determinants. Let an n-rowed determinant in which the diagonal above the leading one is composed of unit elements and all diagonals above it of zeros be said to be *of class* Z_n.

L_{106}. *If* $A = ||a_{ij}||$ *is of class* Z_n *and*

$$|a_{ij}| \leqslant K \qquad (j = 1, 2, \dots, i;\ i = 1, 2, \dots, n),$$

then

$$|A| \leqslant K(1+K)^{n-1}. \tag{3·15}$$

For, on expanding in terms of the elements of the first row, it is seen that A is equal to

$a_{11} \times$ a determinant of class Z_{n-1} − a determinant of class Z_{n-1}, and (3·15) follows from this by induction. The constant on the right of (3·15) is exact, since

$$\begin{vmatrix} K & 1 & 0 & . \\ -K & K & 1 & . \\ K & -K & K & . \\ . & . & . & . \end{vmatrix} = K(1+K)^{n-1}. \quad (n \text{ rows})$$

Now, if $p_0(z), p_1(z), \ldots$ is a simple set of polynomials, the coefficient of the highest power being always unity, it will be found that

$$\pi_{ij} = \text{a determinant of class } Z_{i-j} \qquad (j=0, 1, 2, \ldots, i-1).$$

Thus (3·14) implies that

$$|\pi_{ij}| \leqslant M(1+M)^{i-j-1} < (1+M)^{i-j}.$$

Take $R > 1+M$, so that

$$M_i(R) \leqslant M(1+R+R^2+\ldots+R^{i-1}) + R^i < (i+1)R^i,$$

and $$\omega_n(R) < \sum_{i=0}^{n-1} (i+1) R^i (1+M)^{n-i} + (n+1) R^n$$

$$< \sum_{i=0}^{n-1} (i+1) R^n + (n+1) R^n = \tfrac{1}{2}(n+1)(n+2) R^n.$$

Thus, by L_{102}, if $f(z) = \Sigma a_n z^n$ is such that

$$\Sigma |a_n| (n+1)(n+2) R^n$$

converges, the basic series converges absolutely and uniformly to $f(z)$ in the circle $|z| \leqslant R$. The following result has therefore been established (**11**):

Theorem 2. *Let*

(3·16) $$p_i(z) = p_{i0} + p_{i1}z + \ldots + p_{i,i-1}z^{i-1} + z^i \qquad (i=0, 1, 2, \ldots)$$

be a simple set of polynomials whose coefficients satisfy the inequality

(3·17) $$|p_{ij}| \leqslant M,$$

and let $f(z)$ be regular in $|z| < R$, where $R > 1+M$. Then the basic series

(3·18) $$p_0(z)\,\Pi_0 f(0) + p_1(z)\,\Pi_1 f(0) + \ldots$$

converges absolutely to $f(z)$ in $|z| < R$.

The convergence is uniform in any smaller circle. The theorem is "best possible" in that the condition $R > 1+M$ is essential. For, if

$$p_0(z) = 1, \quad p_i(z) = z^i + M(z^{i-1} - z^{i-2} + z^{i-3} - \dots) \qquad (i \geqslant 1),$$

it will be found that

$$\pi_{ij} = (-)^{i-j} M(1+M)^{i-j-1} \qquad (j = 0, 1, 2, \dots, i-1),$$

and, if $f(z)$ is taken to be $(z+1+M)^{-1}$, the series defining $\Pi_0 f(0)$, namely

$$\frac{1}{1+M} + \frac{M}{(1+M)^2} + \frac{M}{(1+M)^2} + \dots,$$

does not converge. That is to say, there exists a set of polynomials satisfying (3·17) and a function $f(z)$, regular in $|z| < 1+M$, for which (3·18) does not converge.

Uniqueness theorems. In conclusion it is important to point out that in general there is no "uniqueness theorem" associated with a basic series, i.e. the fact that

$$A_0 p_0(z) + A_1 p_1(z) + \dots$$

converges uniformly to zero in some region does not always imply that all the A's are zero, e.g.

$$1 - (z+1) + \left(\frac{z^2}{2!} + z\right) - \left(\frac{z^3}{3!} + \frac{z^2}{2!}\right) + \dots = 0 \quad \text{(all } z\text{)}.$$

This is connected with the fact that multiplication of infinite matrices is not always associative, so that, though a row-finite matrix can have only one row-finite reciprocal, it can also have one-sided reciprocals which are not row-finite. Necessary and sufficient conditions for the "uniqueness theorem" to hold good can be found by applying the argument of Vitali's theorem (**13**).

CHAPTER II

THE SUM OF A FUNCTION

§ 4. Bernoulli polynomials.

If two functions are connected by the relation

$$\Delta g(z) = g(z+1) - g(z) = f(z), \tag{4·1}$$

$f(z)$ is called the *difference* of $g(z)$, and $g(z)$ the *sum* of $f(z)$. The sum is analogous to the integral, but whereas the integral is indeterminate only to the extent of an arbitrary constant, any function of period unity can be added to the sum.

Let us first try to find a polynomial $\phi_n(z)$ such that

$$\phi_n(z+1) - \phi_n(z) = nz^{n-1}. \tag{4·2}$$

The indetermination can be removed by assuming that $\phi_n(z)$ vanishes at the origin. These conditions are evidently equivalent to

$$\begin{cases} \phi_n(0) = 0, & \\ \phi_n^{(s)}(1) - \phi_n^{(s)}(0) = 0 & (s = 0, 1, \ldots, n-2, n, n+1, \ldots), \\ \qquad\qquad = n! & (s = n-1), \end{cases} \tag{4·3}$$

so that 1, $\phi_1(z)$, $\phi_2(z)$, ... are the basic polynomials associated with the simple set of operators

$$f(0),\quad f(1) - f(0),\quad \frac{1}{2!}\{f'(1) - f'(0)\},\quad \frac{1}{3!}\{f''(1) - f''(0)\},\quad \ldots, \tag{4·4}$$

and any polynomial $f(z)$ can be expanded in the series

$$f(z) = f(0) + \{f(1) - f(0)\}\phi_1(z) + \frac{1}{2!}\{f'(1) - f'(0)\}\phi_2(z) + \ldots. \tag{4·5}$$

$\phi_n(z)$ is generally called the nth *Bernoulli polynomial*, but the name is sometimes applied to the slightly different polynomial $B_n(z)$ defined below.

Take $f(z)$ in (4·5) to be $\phi_n'(z)$ and use (4·3);

(4·6) $$\phi_n'(z) = \phi_n'(0) + n\phi_{n-1}(z) \qquad (n = 2, 3, \ldots)$$

Now write

(4·7) $$B_{n-1} = \frac{1}{n}\phi_n'(0), \quad B_0(z) = 1, \quad B_n(z) = \phi_n(z) + B_n \quad (n \geqslant 1)$$

so that (4·2), (4·3), (4·6) give

(4·8) $$B_n(z+1) - B_n(z) = nz^{n-1},$$

(4·9) $$B_n(0) = B_n,$$

(4·10) $$B_n'(z) = nB_{n-1}(z).$$

It follows from the last relation that

(4·11) $$\frac{1}{s!}B_n^{(s)}(0) = \binom{n}{s} B_{n-s} \qquad (0 \leqslant s \leqslant n),$$

so that

(4·12) $$B_n(z) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s} z^s.$$

Again, setting $z = 0$ in (4·8), and $z = 1$ in (4·12),

$$B_1(1) = B_1(0) + 1 = B_1 + 1,$$

(4·13) $$B_n(1) = B_n(0) \qquad (n > 1),$$

$$B_n(1) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s}.$$

Thus

(4·14) $$\begin{cases} B_0 = 1, \\ B_n = \displaystyle\sum_{s=0}^{n} \binom{n}{s} B_s \qquad (n > 1). \end{cases}$$

These are the recurrence relations for the *Bernoulli numbers*.*

The series $$\sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!}$$

* They are sometimes defined differently, by the equation

$$t(e^t - 1)^{-1} = 1 - \frac{t}{2} + \frac{B_1 t^2}{2!} - \frac{B_2 t^4}{4!} + \ldots. \quad \text{Cf. (4·15).}$$

is called the generating function of the Bernoulli polynomials. Now, formally,

$$\sum_{n=0}^{\infty} B_n(z)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}\binom{n}{s}B_s z^{n-s}\right\}\frac{t^n}{n!}$$

$$=\sum_{s=0}^{\infty}\frac{B_s}{s!}t^s\sum_{n=s}^{\infty}\frac{z^{n-s}t^{n-s}}{(n-s)!}=e^{zt}\sum_{s=0}^{\infty}\frac{B_s t^s}{s!}.$$

In this set $z=1$;

$$t+\sum_{n=0}^{\infty}\frac{B_n t^n}{n!}=e^t\sum_{s=0}^{\infty}\frac{B_s t^s}{s!},$$

whence

(4·15) $$\sum_{n=0}^{\infty}\frac{B_n t^n}{n!}=\frac{t}{e^t-1},$$

and

(4·16) $$\sum_{n=0}^{\infty}B_n(z)\frac{t^n}{n!}=\frac{te^{zt}}{e^t-1}.$$

The fact that the expression on the right can be expanded as a power series in t, if $|t|<2\pi$, justifies these operations. Moreover

$$\sum_{n=0}^{\infty}B_n(1-z)\frac{t^n}{n!}=\frac{te^{t-zt}}{e^t-1}=\frac{te^{-zt}}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n(z)\frac{(-t)^n}{n!},$$

so that

(4·17) $$B_n(1-z)=(-)^n B_n(z).$$

Again,

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{m-1}B_n\left(z+\frac{k}{m}\right)\right\}\frac{t^n}{n!}=\frac{t}{e^t-1}\sum_{k=0}^{m-1}e^{\left(z+\frac{k}{m}\right)t}=\frac{te^{zt}}{e^{t/m}-1}$$

$$=m\sum_{n=0}^{\infty}B_n(mz)\left(\frac{t}{m}\right)^n,$$

whence

(4·18) $$\sum_{k=0}^{m-1}B_n\left(z+\frac{k}{m}\right)=m^{1-n}B_n(mz).$$

Various special values of the Bernoulli polynomials, which will be useful later, follow from these relations; e.g.

(4·19) $$B_{2n+1}(1)=B_{2n+1}=0 \qquad (n>0),$$

(4·20) $$B_n(\tfrac{1}{2})=(2^{1-n}-1)B_n.$$

The integrals

$$\int_0^1 B_n(x)\cos 2s\pi x\,dx, \qquad \int_0^1 B_n(x)\sin 2s\pi x\,dx$$

can be evaluated by integrating by parts, using (4·10), (4·13), (4·19), and yield the Fourier series, valid for $0 \leqslant x \leqslant 1$,

$$(4\cdot21)\qquad \begin{cases} B_{2n}(x) = (-)^{n+1}\dfrac{2(2n)!}{(2\pi)^{2n}}\displaystyle\sum_{s=1}^{\infty}\frac{\cos 2s\pi x}{s^{2n}}, \\ B_{2n+1}(x) = (-)^{n+1}\dfrac{2(2n+1)!}{(2\pi)^{2n+1}}\displaystyle\sum_{s=1}^{\infty}\frac{\sin 2s\pi x}{s^{2n+1}}. \end{cases}$$

Set $x=0$ in the first of these;

$$(4\cdot22)\qquad B_{2n} = (-)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{s=1}^{\infty}\frac{1}{s^{2n}} \qquad (n>0).$$

The function obtained by subtracting from $B_n(z)$ the first s terms of its Fourier series was investigated by Hurwitz (**1**), and is of considerable importance in the sequel. Consider the integral

$$(4\cdot23)\qquad B_{n,s}(z) = \frac{n!}{2\pi i}\int_{C_s}\frac{e^{tz}\,t^{-n}}{e^t-1}\,dt,$$

where C_s is the circle $|t| = (2s+1)\pi$ and s is a positive integer. Evidently

$$(4\cdot24)\qquad B_{n,0}(z) = B_n(z),$$

and, by the residue theorem,

$$(4\cdot25)\qquad B_{n,s}(z) - B_n(z) = n!\sum_{k=1}^{s}\frac{e^{2k\pi iz} + (-)^n e^{-2k\pi iz}}{(2k\pi i)^n}.$$

Thus, for all values of n and s,

$$(4\cdot26)\qquad B_{n,s}(z+1) - B_{n,s}(z) = nz^{n-1}.$$

It is easy to see* that $(e^t-1)^{-1}$ is uniformly bounded on the circles C_s, so that

$$(4\cdot27)\qquad |B_{n,s}(z)| \leqslant \frac{n!}{2\pi}\int_0^{2\pi}\frac{e^{(2s+1)\pi|z|}(2s+1)\pi}{|e^t-1|\{(2s+1)\pi\}^n}\,d\theta$$
$$< A\,n!\,e^{(2s+1)\pi|z|}\{(2s+1)\pi\}^{1-n},$$

when A is an absolute constant.

These results enable us to determine the order and type of the Bernoulli polynomials.

* Cf. Dienes (**1**), 252.

L_{201}. $1, \phi_1(z), \phi_2(z), \ldots$ *and* $B_0(z), B_1(z), \ldots$ *are both sets of order* 1, *type* $1/2\pi$.

For

$$(4{\cdot}28) \quad B_n(z) = -n!\frac{e^{2\pi iz} + (-)^n e^{-2\pi iz}}{(2\pi i)^n} + O\left(\frac{n!\,e^{3\pi|z|}}{(3\pi)^{n-1}}\right),$$

so that the maximum modulus of $\phi_n(z)$ satisfies the inequality

$$\Phi_n(R) < K(R)\,n!\,(2\pi)^{-n}.$$

Thus

$$\omega_n(R) = \sum_{i=1}^{n} \frac{n!}{i!\,(n-i+1)!}\Phi_i(R) < Kn! \sum_{i=1}^{n} \frac{1}{(n-i+1)!}(2\pi)^{-i}$$

$$= Kn!\,(2\pi)^{-n} \sum_{s=0}^{n-1} \frac{(2\pi)^s}{(s+1)!} < Kn!\,(2\pi)^{-n},$$

and so

$$\omega = \omega(R) \leqslant 1,$$

$$\gamma = \gamma(R) \leqslant \frac{1}{2\pi}.$$

On the other hand the series (4·5) vanishes identically when $f(z) = \sin 2\pi z$, so, by L_{105}, the increase of $1, \phi_1(z), \phi_2(z), \ldots$ cannot be less than order 1, type $1/2\pi$. The set $B_0(z), B_1(z), \ldots$ can be treated in the same way.

§ 5. The sum of a function.

Integral functions. In virtue of (4·8) a polynomial

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n$$

has the sum

$$(5{\cdot}1) \qquad g(z) = a_0 B_1(z) + \frac{a_1}{2} B_2(z) + \ldots + \frac{a_n}{n+1} B_{n+1}(z).$$

In the case of an integral function $f(z) = \Sigma a_n z^n$ the series corresponding to (5·1) does not always converge. In place of it we consider this series

$$(5{\cdot}2) \qquad g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} B_{n+1, s_n}(z)$$

and endeavour to choose $s_0, s_1, \ldots$ in such a way that this series converges uniformly in any finite region of the plane. This is always possible. For example, take $s_n = n$, so that, by (4·27),

$$|B_{n, s_n}(z)| < A\,n!\,e^{(2n+1)\pi|z|}\{(2n+1)\pi\}^{1-n}$$

$$< Ae^{(2n+1)\pi|z|},$$

and $g(z)$, defined by (5·2), is an integral function. The result thus established, that it is always possible to find an integral function which is the sum of a given integral function, is due to Guichard (**1**), the proof given above to Appell (**1**) and Hurwitz (**1**).

It is possible to prove a more precise result (**7, 12**).

Theorem 3. *If $f(z)$ is an integral function, there is an integral function $g(z)$, of the same order as $f(z)$, such that*

$$g(z+1)-g(z)=f(z).$$

In the case of a function of infinite order this reduces to Guichard's theorem, so we may suppose that ρ, the order of $f(z)$, is finite. If $\rho<1$ it is sufficient to take

$$g(z)=\sum_{n=0}^{\infty}\frac{a_n}{n+1}B_{n+1}(z).$$

For, by (4·27),

$$|B_n(z)|<A\,n!\,e^{\pi|z|}\pi^{1-n}<n^n e^{\pi|z|} \qquad (n\geqslant n_0),$$

and, if $\rho^{-1}>\lambda>1$, $\qquad |a_n|<n^{-n\lambda} \qquad (n\geqslant n_\lambda).$

It follows that the series defining $g(z)$ converges and that it can be differentiated any number of times. Thus

$$g^{(k)}(0)=\sum_{n=k-1}^{\infty}\frac{a_n}{n+1}B_{n+1}^{(k)}(0)$$

$$=\sum_{n=k-1}^{\infty}a_n n(n-1)\dots(n-k+2)\,B_{n-k+1}.$$

Now it is clear from (4·19), (4·22) that

$$(5·3) \qquad |B_n|\leqslant 4n!\,(2\pi)^{-n} \qquad (n\geqslant 0),$$

so that $\qquad |g^{(k)}(0)|\leqslant 4\sum_{n=k-1}^{\infty}n^{-n\lambda}n!\,(2\pi)^{k-n-1} \qquad (k\geqslant k_\lambda).$

Moreover

$$\frac{n^{-n\lambda}n!\,(2\pi)^{k-n-1}}{(n+1)^{-(n+1)\lambda}(n+1)!\,(2\pi)^{k-n-2}}$$

$$=2\pi\left(1+\frac{1}{n}\right)^{n\lambda}(n+1)^{\lambda-1}>k^{\lambda-1} \qquad (n\geqslant k-1),$$

so that

$$|g_k| = \frac{1}{k!}|g_k(0)| \leqslant \frac{4(k-1)^{-\lambda(k-1)}(k-1)!}{k!} \sum_{s=0}^{\infty} k^{s(1-\lambda)} < (k-1)^{-\lambda(k-1)}$$

$$(k \geqslant k_\lambda).$$

Thus $$\varliminf_{k\to\infty} \frac{\log|g_k|^{-1}}{k\log k} \geqslant \lambda.$$

This is true for every $\lambda < 1/\rho$, so that the order of $g(z)$ cannot be greater than ρ. On the other hand it is obvious that no sum of $f(z)$ can be of order less than ρ.

If $\rho \geqslant 1$, consider the function $\lambda(n)$ defined by

$$(5\cdot4) \qquad \lambda(n) = \min_{m\geqslant n}\left(\frac{\log|a_m|^{-1}}{m\log m}\right).$$

$\lambda(n)$ is an increasing function, and

$$(5\cdot5) \qquad \lambda(n) \to \varliminf_{n\to\infty} \frac{\log|a_n|^{-1}}{n\log n} = \frac{1}{\rho}.$$

Again, writing $\alpha(n) = 1 - \lambda(n)$,

$$(5\cdot6) \qquad n^{n\alpha(n)}\{(2s_n+1)\pi\}^{-n} < e^{-n} \qquad (n \geqslant 1),$$

provided that $$(2s_n+1)\pi > en^{\alpha(n)},$$

and this is so if

$$(5\cdot7) \qquad s_n = [n^{\alpha(n)}].$$

It will be shown that with this choice of $s_0, s_1, \ldots$, (5·2) defines an integral function of order ρ.

Corresponding to a given value of $r = |z|$ define the integer

$$(5\cdot8) \qquad N(r) = [(8r)^{1/\lambda(2)}].$$

Then (4·27), (5·4), (5·6), (5·7) show that

$$v_n(z) = \left|\frac{a_n B_{n+1,s_n}(z)}{n+1}\right| < n^{-n\lambda(n)} n^n \{(2s_n+1)\pi\}^{-n} \exp\{(2s_n+1)\pi r\}$$

$$(5\cdot9) \qquad < \exp\{-n + 7n^{\alpha(n)} r\} < e^{-n/8} \qquad (n > N),$$

since

$$n - 7n^{\alpha(n)}r - \frac{n}{8} > \tfrac{7}{8}n - \tfrac{7}{8}n^{\alpha(n)+\lambda(2)} = \tfrac{7}{8}n\{1 - n^{\lambda(2)-\lambda(n)}\} \geqslant 0.$$

Now take a fixed integer p and write $\alpha(p)=\alpha$. Then

$$(5\cdot 10)\qquad \sum_{n=p}^{N} v_n(z) < \sum_{n=p}^{N} \exp\{-n+7n^{\alpha(n)}r\}$$
$$\leqslant \sum_{n=p}^{N} \exp\{-n+7n^{\alpha}r\} < N(r)\,\mu(r),$$

where $\mu(r)$ is the greatest term of the last series. $\mu(r)$ is determined by finding the maximum value of the function

$$-x+7x^{\alpha}r,$$

namely
$$\left(\frac{1}{\alpha}-1\right)(7\alpha r)^{1/(1-\alpha)},$$

so that

$$(5\cdot 11)\qquad \mu(r)\leqslant \exp\left\{\left(\frac{1}{\alpha}-1\right)(7\alpha r)^{1/(1-\alpha)}\right\}.$$

Now, making use of (5·9), (5·10),

$$M_0(r)=\max_{|z|=r}|g(z)| < \sum_{n=0}^{p-1} v_n(z)+N(r)\,\mu(r)+\sum_{n=N+1}^{\infty} e^{-n/8},$$

whence, by (5·8), (5·11),

$$\varlimsup_{r\to\infty}\frac{\log\log M_0(r)}{\log r}\leqslant(1-\alpha)^{-1}=\{\lambda(p)\}^{-1}.$$

Finally, make p tend to infinity, and use (5·5).

It follows that $g(z)$ is of order less than or equal to ρ, and, as before, its order cannot be less than ρ.

Meromorphic functions. In the paper referred to, Hurwitz extended Guichard's theorem to meromorphic functions, that is to say, he proved the following result:

L_{202}. *If $f(z)$ is any meromorphic function, there is a meromorphic function $g(z)$ such that*

$$g(z+1)-g(z)=f(z).$$

This theorem can also be made more precise (**8, 14**), but the matter is not so simple as in the case of integral functions. For the order of $g(z)$ is in general not ρ but $\rho+1$. This is inevitable. It is due to the fact that, whereas the sum of a polynomial is a polynomial, the sum of a rational function is in general a meromorphic function of order greater than or equal to one.

Theorem 4. *If $f(z)$ is a meromorphic function of order ρ, there is a meromorphic function $g(z)$, of order less than or equal to $\rho+1$, such that*

$$g(z+1)-g(z)=f(z).$$

Suppose, to start with, that ρ is finite and that the poles b_1, b_2, ... of $f(z)$ are all in $\mathbf{R}z \leqslant 0$. Define $\rho(k)$ for integral values of k by the equation

$$\max_{|z-k|=k-1} \log^+ |f(z)| = k^{\rho(k)}, \tag{5·12}$$

and write

$$\lambda(k) = \max_{l \geqslant k} \rho(l) \geqslant \rho(k), \tag{5·13}$$

so that $\lambda(k)$ is a decreasing function. It will be shown that

$$\lambda = \lim_{k \to \infty} \lambda(k) \leqslant \rho. \tag{5·14}$$

In the z-plane draw the circle $|z| = 2k$ and the line $\mathbf{R}z = 1$, and denote by C the closed contour consisting of the part of the line intercepted by the circle and the arc of the circle to the right of it. Then, if $\rho < \alpha$,

$$M_k = \max_C |f(z)| < \exp k^\alpha \qquad (k \geqslant k_\alpha). \tag{5·15}$$

For,* if $z = re^{i\theta}$ and $R > r$,

$$\log^+ |f(re^{i\theta})| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\phi})| \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\phi-\theta)} d\phi + \sum_{|b_s|<R} \log \left| \frac{R^2-\bar{b}_s z}{R(z-b_s)} \right|.$$

If z is on C, $\qquad |R^2 - \bar{b}_s z| \leqslant 2R^2, \quad |z-b_s| \geqslant 1,$

so that, in the usual Nevanlinna notation,

$$\log^+ |f(re^{i\theta})| \leqslant \frac{R+r}{R-r} m(R,\infty) + n(R,\infty)\log 2R.$$

If $\rho < \beta < \alpha$, $\qquad m(R,\infty),\ n(R,\infty) < R^\beta \qquad (R \geqslant R_\beta)$,

so that, taking $R = 3k$,

$$\log^+ M_k \leqslant 5m(3k,\infty) + n(3k,\infty)\log 6k < k^\alpha \quad (k \geqslant k_\alpha),$$

which is (5·15).

* Nevanlinna (**1**), 25.

Now the circle $|z-k|=k-1$ is enclosed by C, so that, for all k,

$$k^{\rho(k)} \leqslant \log^+ M_k,$$

and hence $$\rho(k) < \alpha \qquad (k \geqslant k_\alpha).$$

Thus $$\lambda = \varlimsup_{k\to\infty} \rho(k) \leqslant \alpha,$$

and, since α may be any number greater than ρ, this gives (5·14).

Next consider the polynomials

$$P_k(z) = \sum_{n=0}^{p_k} \frac{f^{(n)}(k)}{n!} z^n \qquad (k \geqslant 1),$$

where p_k is to be chosen so that

(5·16) $$|P_k(z) - f(z+k)| < \frac{1}{k^2} \qquad (|z| \leqslant \tfrac{1}{3}(k-1)).$$

Cauchy's inequality gives, for $n \geqslant 0$,

$$\frac{f^{(n)}(k)}{n!}(k-1)^n \leqslant \max_{|z-k|=k-1} |f(z)| \leqslant \exp k^{\lambda(k)},$$

so that, if $|z| \leqslant \frac{1}{3}(k-1)$,

$$\begin{aligned} |P_k(z) - f(z+k)| &\leqslant \sum_{n=p_k+1}^{\infty} \frac{|f^{(n)}(k)|}{k!} \left(\frac{k-1}{3}\right)^n \\ &\leqslant \tfrac{1}{2}\, 3^{-p_k} \exp k^{\lambda(k)} < k^{-2}, \end{aligned}$$

provided that*

(5·17) $$p_k = [k^{\lambda(k)}] + 1.$$

(5·16) shows that the series

$$F(z) = \sum_{k=0}^{\infty} \{P_k(z) - f(z+k)\} \qquad (P_0(z) = 0)$$

converges uniformly in any finite region of the plane, neglecting a finite number of terms at the beginning. $F(z)$ is therefore a meromorphic function. It will be shown that its order does not exceed $\rho + 1$.

* There are trivial modifications if $\lambda = 0$. Take

$$p_k = \max\{[k^{\lambda(k)}] + 1,\ 100 \log k\}.$$

Take a fixed integer l and, corresponding to a given value of $r=|z|$, define $q_r=[3r]+1$. The inequality*

$$m(r,f_1+f_2+\ldots+f_q)\leqslant\sum_1^q m(r,f_s)+\log q$$

gives

$$(5\cdot18)\quad m(r,F)\leqslant m\left\{r,\sum_{k=0}^{l}\{P_k(z)-f(z+k)\}\right\}+\sum_{k=l+1}^{q_r} m(r,P_k(z))$$
$$+\sum_{k=l+1}^{q_r} m(r,f(z+k))+m\left\{r,\sum_{k=q_r+1}^{\infty}\{P_k(z)-f(z+k)\}\right\}$$
$$+\log(2q_r+2).$$

Now, if $l+1\leqslant k\leqslant q_r$, $r\geqslant 100$,

$$m(r,P_k(z))\leqslant\log^+\sum_{n=0}^{p_k}\frac{|f^{(n)}(k)|\,r^n}{n!}\leqslant\log^+\sum_{n=0}^{p_k}\left(\frac{r}{k-1}\right)^n\exp k^{\lambda(l)}$$
$$<\log p_k+p_k\log r+k^{\lambda(l)}<2q_r^{\lambda(l)}\log r,$$

and so

$$\sum_{k=l+1}^{q_r} m(r,P_k(z))<2q_r^{\lambda(l)+1}\log r<2(4r)^{\lambda(l)+1}\log r.$$

Next, $f(z)$ is of the form $f_1(z)/f_2(z)$, where $f_1(z), f_2(z)$ are integral functions of order less than or equal to ρ. For the same range of k, the inequality
$$m(r,fg)\leqslant m(r,f)+m(r,g)$$
and Nevanlinna's form of Jensen's theorem† give

$$m(r,f(z+k))\leqslant m(r,f_1(z+k))+m\left(r,\frac{1}{f_2(z+k)}\right)$$
$$=m(r,f_1(z+k))+m(r,f_2(z+k))-N\left(r,\frac{1}{f_2(z+k)}\right)-\log|c_\lambda|$$
$$\leqslant m(r,f_1(z+k))+m(r,f_2(z+k))-\log|c_\lambda|$$
$$\leqslant\log M_1(4r+1)+\log M_2(4r+1)-\log|c_\lambda|$$
$$<r^{\rho+\epsilon}\qquad(r\geqslant r_\epsilon),$$

where c_λ is a constant, $M_1(r)$ and $M_2(r)$ are the maxima of $|f_1(z)|$ and $|f_2(z)|$ for $|z|=r$, and ϵ is a given positive number.

Again, by (5·16),

$$m\left\{r,\sum_{k=q_r+1}^{\infty}\{P_k(z)-f(z+k)\}\right\}<\log\sum_1^\infty\frac{1}{k^2}.$$

* Nevanlinna (1), 14.

† Nevanlinna (1), 14, 6.

Thus, as $r \to \infty$, the five terms on the right of (5·18) are respectively $O(r^{\rho+\epsilon})$, $O(r^{\lambda(l)+1}\log r)$, $O(r^{\rho+\epsilon+1})$, $O(1)$, $O(\log r)$.

Again, the poles of $F(z)$ are the points $b_n - m$ $(n \geqslant 1, m \geqslant 0)$, so that (remembering that all the b's are to the left of the imaginary axis)

$$n\{r, \infty, F(z)\} \leqslant rn\{r, \infty, f(z)\} < r^{\rho+\epsilon+1} \qquad (r \geqslant r_\epsilon').$$

Hence
$$T(r, F) = m(r, F) + N(r, F)$$

consists of the five terms enumerated above together with a term $O(r^{\rho+\epsilon+1})$, and so

$$\overline{\lim_{r\to\infty}} \frac{\log T(r, F)}{\log r} \leqslant \max\{\lambda(l)+1, \rho+\epsilon+1\}.$$

Since $\lambda(l) \to \lambda \leqslant \rho$ and ϵ is arbitrary it follows that the order of $F(z)$ cannot exceed $\rho + 1$.

Now, if N is any integer,

$$\Delta F(z) - f(z) = \sum_{k=0}^{N} \{P_k(z+1) - P_k(z)\} - f(z+N+1)$$
$$+ \sum_{k=N+1}^{\infty} \{P_k(z+1) - f(z+k+1)\} - \sum_{k=N+1}^{\infty} \{P_k(z) - f(z+k)\},$$

and, if z is confined to any given finite region of the plane, the functions on the right are regular for sufficiently large values of N. Hence

$$\Delta F(z) - f(z) = h(z),$$

an integral function. Evidently the order of $h(z)$ cannot exceed $\rho+1$. Thus, by Theorem 3, there is an integral function $H(z)$ of order less than or equal to $\rho + 1$ such that

$$\Delta H(z) = h(z),$$

and so
$$g(z) = F(z) - H(z)$$

is a meromorphic function of order less than or equal to $\rho + 1$ satisfying (4·1).

If ρ is infinite, or if we only want to prove that a meromorphic function has a sum without enquiring about its order, the argument is much simpler. Choose $p_1, p_2, \ldots$ in any way such that (5·16) is satisfied. Then, as before, $\Delta F(z) - f(z)$ is an integral function and so is the difference of an integral function $H(z)$, and $g(z) = F(z) - H(z)$ is the required solution of (4·1).

If the poles of $f(z)$ are in $\mathbf{R}z \leqslant d$, the theorem is proved by applying the result of the preceding section to $f(z+d)$. If they are in $\mathbf{R}z \geqslant d$, the poles of $f(-z)$ are in $\mathbf{R}z \leqslant -d$, and so there is a function $g_1(z)$ of order less than or equal to $\rho+1$, satisfying

$$g_1(z+1)-g_1(z)=f(-z),$$

and so
$$g(z)=-g_1(-z+1)$$

satisfies (4·1).

In the general case, when the poles of $f(z)$ may be in any part of the plane, we make use of Theorem A (p. 2). The projections of the nebulae on the real axis will form a set of finite measure. Let d be a real number which does not belong to it. Then the line $\mathbf{R}z=d$ does not intersect any nebula and so it divides the pole clusters into two groups, one on each side. The expansion of $f(z)$ in normal form will divide into two parts corresponding to these (the integral function being assigned to either) so that

$$f(z)=L_d(z)+R_d(z),$$

where $L_d(z)$, $R_d(z)$ are both of order less than or equal to ρ, the poles of $L_d(z)$ being in $\mathbf{R}z<d$ and those of $R_d(z)$ in $\mathbf{R}z>d$. By what has been proved, both functions can be expressed as differences of meromorphic functions of order not exceeding $\rho+1$, and hence their sum can be so expressed.

The number $\rho+1$ *in Theorem* 4 *is "best possible"*. For let $a_1, a_2, \ldots$ be an increasing sequence of positive numbers with exponent of convergence ρ, satisfying the condition that no two a's differ by an integer, and let $f(z)$ be a meromorphic function of order ρ with poles at these points. It is easy to see that $g(z)$ must have poles either at all points a_n-m+1 $(m \geqslant 1)$, or else at all points a_n+m $(m \geqslant 1)$, according as a_n is taken to be a pole of $g(z)$ or of $g(z+1)$. Thus the order of $g(z)$ cannot be less than the exponent of convergence of the double sequence a_n+m $(n, m \geqslant 1)$. Now, if $\lambda>0$,

$$\sum_{n,\,m=1}^{\infty}(a_n+m)^{-\lambda} > \sum_{n=1}^{\infty}\int_2^{\infty}(a_n+x)^{-\lambda}dx > K\sum_{n=1}^{\infty}a_n^{1-\lambda},$$

so that for convergence we must have $\lambda \geqslant \rho+1$.

§ 6. Linear difference equations.

The equation

$$(6{\cdot}1) \qquad g(z+1)=f(z)\,g(z),$$

where $f(z)$ is a given integral or meromorphic function, can be solved by means of the preceding theorem.

$f(z)$ is of the form $f_1(z)\,f_2(z)$, the poles and zeros of $f_1(z)$ being in $\mathbf{R}z \leqslant 0$, and those of $f_2(z)$ in $\mathbf{R}z > 0$.

$f_1'(z)/f_1(z)$ is a meromorphic function, of order less than or equal to ρ, with simple poles in $\mathbf{R}z \leqslant 0$, the residues at the poles being positive or negative integers. By Theorem 4 there is a function $h(z)$, of order less than or equal to $\rho+1$, such that

$$h(z+1)-h(z)=\frac{f_1'(z)}{f_1(z)}.$$

The method of constructing $h(z)$ ensures that its poles are also simple and the residues at them positive or negative integers. Hence, by integration,

$$h(z)=\frac{H'(z)}{H(z)},$$

where $$H(z)=e^{F(z)}\frac{C_1(z)}{C_2(z)},$$

$F(z)$ being an integral function and $C_1(z)$, $C_2(z)$ canonical products. $C_1(z)$, $C_2(z)$ are of order less than or equal to $\rho+1$. For if one of them was of order greater than $\rho+1$, the exponent of convergence of its zeros, and hence the exponent of convergence of the poles of $h(z)$, would be greater than $\rho+1$. This is impossible. Now

$$f_1(z)=\exp\left\{\int\{h(z+1)-h(z)\}\,dz\right\}$$

$$=\frac{H(z+1)}{H(z)}=e^{F(z+1)-F(z)}\frac{C_1(z+1)\,C_2(z)}{C_1(z)\,C_2(z+1)},$$

and so $F(z+1)-F(z)$ is a polynomial of degree less than or equal to ρ. Thus

$$F(z+1)-F(z)=P(z+1)-P(z),$$

where $P(z)$ is a polynomial of degree less than or equal to $\rho + 1$, and

$$g_1(z) = e^{P(z)} \frac{C_1(z)}{C_2(z)}$$

is a function of order less than or equal to $\rho + 1$, such that

$$g_1(z+1) = f_1(z) g_1(z).$$

$f_2(z)$ can be treated in the same way, and $g(z)$ is then the product of $g_1(z)$, $g_2(z)$.

The following result has therefore been established (**8**):

Theorem 5. *If $f(z)$ is an integral or meromorphic function of order ρ, there is a meromorphic function $g(z)$, of order less than or equal to $\rho + 1$, such that*

$$g(z+1) = f(z) g(z).$$

As before, the number $\rho + 1$ is "best possible". For example, the equation

$$g(z+1) = zg(z)$$

has no solution of lower order than $\Gamma(z)$.

Various other difference equations can be reduced to the fundamental types (4·1), (6·1), e.g. the general linear difference equation of the first order

$$P(z) g(z+1) + Q(z) g(z) = R(z), \tag{6·2}$$

where $P(z)$, $Q(z)$, $R(z)$ are given integral or meromorphic functions, which can be solved in the following way.

Let $n(z)$ be a solution of the equation

$$n(z+1) = -\frac{Q(z)}{P(z)} n(z), \tag{6·3}$$

and let

$$g(z) = n(z) h(z).$$

Then (6·2) is

$$P(z) n(z+1) h(z+1) + Q(z) n(z) h(z) = R(z),$$

or

$$h(z+1) - h(z) = -\frac{R(z)}{Q(z) n(z)}. \tag{6·4}$$

Thus (6·2) is equivalent to (6·3), (6·4).

CHAPTER III

PROPERTIES OF SUCCESSIVE DERIVATIVES

§ 7. A theorem of Pólya.

The operators $f(a_0), f'(a_1), f''(a_2), \ldots$, where $a_0, a_1, \ldots$ is any sequence of numbers, evidently form a simple set. Recently Gontcharoff, Takenaka and Kakeya have discussed the associated basic series

$$(7{\cdot}1) \qquad G_0(z) f(a_0) + \frac{G_1(z)}{1!} f'(a_1) + \frac{G_2(z)}{2!} f''(a_2) + \ldots,$$

and have deduced a number of interesting properties of successive derivatives. As a preliminary we prove a beautiful theorem of Pólya (**4**).

Let $f(z)$ be a meromorphic function, a one of its poles. Let the domain consisting of those points z which are nearer to a than to any other pole be called "the county of a" or "a-shire". It is evident that, if a-shire and b-shire have boundary points in common, the common boundary is the line bisecting at right angles the line joining a, b. Thus a county consists of the interior of a convex polygon, which may be finite and have a finite number of sides, or be infinite and have either a finite or an infinite number of sides. Thus in the case of $\wp(z)$ the counties are all congruent, and each is in general a hexagon with three pairs of parallel sides. Now consider the set E consisting of all the zeros of $f(z)$, $f'(z)$, $f''(z)$, ... taken together. Then Pólya's theorem asserts that the derived set E' is identical with the "map", i.e. the aggregate of county boundaries defined above. Thus:

Theorem 6. *Let $f(z)$ be a meromorphic function and let E denote the set of zeros of $f(z)$, $f'(z)$, ... Then a point z belongs to E' if and only if it is equidistant from the two poles which are nearest to it.*

It follows that E' depends only on the position of the poles, not, for example, on their multiplicity.

Consider the sequence

$$(7\cdot 2) \qquad \left|\frac{f'(z)}{1!}\right|, \quad \left|\frac{f''(z)}{2!}\right|^{\frac{1}{2}}, \ldots,$$

$f(z)$ being a meromorphic function, and denote by $[R, \delta]$ the closed region consisting of the circle $|z| \leqslant R$, less those points whose distance from a pole of $f(z)$ is less than δ.

$\mathbf{L}_{301}$. (i) *The sequence* (7·2) *is uniformly bounded in every region* $[R, \delta]$.

(ii) *If z belongs to a-shire,*

$$\left|\frac{f^{(n)}(z)}{n!}\right|^{1/n} \to \frac{1}{|z-a|}.$$

Moreover the convergence is uniform in any closed part of a-shire.

To prove (i), consider the region $[R+\frac{1}{2}\delta, \frac{1}{2}\delta]$. In this $f(z)$ is regular and so has a maximum modulus M. Again, if a point z of $[R, \delta]$ is surrounded by a circle of radius $\frac{1}{2}\delta$, this circle will lie entirely in $[R+\frac{1}{2}\delta, \frac{1}{2}\delta]$. Hence, by a classical inequality,

$$\left|\frac{f^{(n)}(z)}{n!}\right| \leqslant \frac{M}{(\frac{1}{2}\delta)^n},$$

or

$$\left|\frac{f^{(n)}(z)}{n!}\right|^{1/n} \leqslant \frac{2M^{1/n}}{\delta}.$$

To prove (ii), consider a pole a of multiplicity $q+1$, and write

$$f(z) = \frac{A_0}{z-a} + \frac{1!\,A_1}{(z-a)^2} + \ldots + \frac{q!\,A_q}{(z-a)^{q+1}} + \Phi(z),$$

where $\Phi(z)$ is regular at a and $A_q \neq 0$. Then

$$f^{(n)}(z) = (-)^n \left\{\frac{n!\,A_0}{(z-a)^{n+1}} + \ldots + \frac{(n+q)!\,A_q}{(z-a)^{n+q+1}}\right\} + \Phi^{(n)}(z),$$

and

$$\frac{f^{(n)}(z)}{n!} = (-)^n \frac{(n+1)(n+2)\ldots(n+q)}{(z-a)^{n+q+1}} A_q$$

$$\times \left\{1 + \frac{(z-a)^{q+1}(a-z)^n \Phi^{(n)}(z)}{(n+1)\ldots(n+q)\,n!\,A_q} + \sum_{k=0}^{q-1} \frac{(z-a)^{q-k} A_k}{(n+k+1)\ldots(n+q)\,A_q}\right\}$$

$$= (-)^n \frac{(n+1)\ldots(n+q)}{(z-a)^{n+q+1}} A_q \{1+\phi_n(z)\}.$$

Denote by $\rho(z)$ the radius of convergence of the Taylor series of $\Phi(z)$ at a point z. $\rho(z)$ is a continuous function. For, if $|z'-z|<\rho(z)$, the circle of centre z' and radius $\rho(z)-|z'-z|$ lies in the circle of centre z and radius $\rho(z)$, and hence

$$\rho(z') \geqslant \rho(z) - |z'-z|.$$

Similarly

$$\rho(z') \leqslant \rho(z) + |z'-z|.$$

Now consider the function $|z-a|/\rho(z)$. It is continuous at every point z at which $\rho(z)>0$, and hence at every point of a-shire. Moreover

$$\frac{|z-a|}{\rho(z)} < 1$$

at every point of a-shire, since the nearest singularity of $\Phi(z)$ is the nearest pole of $f(z)$ other than a, and this is more distant than a. Denote by A a closed region lying wholly in the finite part of a-shire, by ρ_0 the minimum of $\rho(z)$ in A, and by α the maximum of $|z-a|/\rho(z)$. Then

$$\rho(z) \geqslant \rho_0 > 0, \quad \frac{|z-a|}{\rho(z)} \leqslant \alpha < 1 \quad \text{in } A. \tag{7·3}$$

Choose β so that $\alpha<\beta<1$, and surround each point z of A by a circle of radius $\beta\rho(z)$. These circles will cover a region $A^\star$, every point of which is distant at least $(1-\beta)\rho_0$ from the nearest singularity of $\Phi(z)$. Hence $\Phi(z)$ is regular inside and on the boundary of $A^\star$, and so

$$|\Phi(z)| \leqslant M' \text{ in } A^\star.$$

Now each point z of A is surrounded by a circle of radius $\beta\rho(z)$ lying wholly in $A^\star$. Hence

$$\left|\frac{\Phi^{(n)}(z)}{n!}\right| \leqslant \frac{M'}{\{\beta\rho(z)\}^n},$$

and so, by (7·3),

$$\left|\frac{(z-a)^n \Phi^{(n)}(z)}{n!}\right| \leqslant \left(\frac{\alpha}{\beta}\right)^n M'.$$

It follows that $\phi_n(z) \to 0$ uniformly in A, and this establishes (ii).

The second part of Theorem 6 is an immediate deduction. For

$$\left|\frac{f^{(n)}(z)}{n!}\right|^{1/n} > \frac{1}{2}\frac{1}{|z-a|} \qquad (n>\nu) \text{ in } A,$$

so that none of $f^{(\nu+1)}(z), f^{(\nu+2)}(z), \ldots$ can have zeros in A. Moreover $f(z), f'(z), \ldots f^{(\nu)}(z)$ have only a finite number of zeros in A. No interior point of A can therefore belong to E' and hence no point of a-shire can.

To prove the first part of the theorem two further lemmas are needed.

$\mathbf{L}_{302}$. *Let $f_1(z), f_2(z), \ldots$ be regular in an open domain B, and let the sequence*

$$\mathbf{R}f_1(z), \quad \mathbf{R}f_2(z), \quad \ldots$$

converge uniformly in every closed part $B^\star$ of B. Moreover, let there be a point z_0 of B such that

$$f_1(z_0), \quad f_2(z_0), \quad \ldots$$

converges. Then the given sequence converges uniformly in every closed part of B.

Take B to be the unit circle and $z_0=0$. The general result can then be deduced by the familiar process of covering with circles. For $B^\star$ take the circle $|z|\leqslant r$ and let $r<R<1$.

Then the hypotheses are that, for given $\epsilon>0$,

$$|f_m(0)-f_n(0)|<\epsilon, \quad |\mathbf{R}f_m(z)-\mathbf{R}f_n(z)|<\epsilon \quad (|z|\leqslant R, m, n>n_0).$$

Borel's inequality* applied to $f_m(z)-f_n(z)-f_m(0)+f_n(0)$ gives

$$|f_m(z)-f_n(z)|<\epsilon+\frac{8R\epsilon}{R-r} \qquad (|z|\leqslant r, m, n>n_0),$$

which is the result stated.

$\mathbf{L}_{303}$. *Let $\mu_1, \mu_2, \ldots$ be a sequence of positive numbers tending to ∞ and let $f_1(z), f_2(z), \ldots$ be regular in an open domain B and such that*

$$(7{\cdot}4) \qquad |f_1(z)|^{1/\mu_1}, \quad |f_2(z)|^{1/\mu_2}, \quad \ldots$$

converges uniformly in every closed part $B^\star$ of B to a limit function never equal to zero. Let z_0 be a given point of B. Then it is possible to choose the determinations of

$$(7{\cdot}5) \qquad f_1(z)^{1/\mu_1}, \quad f_2(z)^{1/\mu_2}, \quad \ldots$$

* Valiron (**1**), 20.

in such a way that this sequence converges in B (and uniformly in every closed part $B^\star$) and the argument of the limit function at z_0 has any given value.

Let α be the assigned value of the argument of the limit function at z_0. Choose the determination of $\log f_n(z)$ to be that for which

$$-\pi+\mu_n\alpha < \mathrm{I}\log f_n(z_0) \leqslant \pi+\mu_n\alpha.$$

This is possible, at any rate for sufficiently large n, since then $f_n(z_0) \neq 0$. It follows that

$$\mathrm{I}\,\frac{\log f_n(z_0)}{\mu_n} \to \alpha. \tag{7·6}$$

The real part of this sequence likewise tends to a limit, by hypothesis, so

$$\frac{\log f_1(z)}{\mu_1}, \; \frac{\log f_2(z)}{\mu_2}, \; \ldots \tag{7·7}$$

is convergent at z_0.

Let $B^\star$ be a closed part of B. As the convergence of (7·4) is uniform in $B^\star$, the limit function must be continuous. As it does not vanish in $B^\star$ it must have a minimum $\delta > 0$.

Again, since the convergence is uniform,

$$|f_n(z)|^{1/\mu_n} > \tfrac{1}{2}\delta \qquad (z \text{ in } B^\star,\ n > n_0),$$

and so $\log f_n(z)$ is regular in $B^\star$, for $n > n_0$. L_{302} now shows that the sequence (7·7) converges in B, and uniformly in any $B^\star$. The same is therefore true of the sequence with general term

$$f_n(z)^{1/\mu_n} = \exp\frac{\log f_n(z)}{\mu_n}.$$

It has already been shown, in (7·6), that the argument of the limit function at z_0 has the assigned value α.

Let C be a circle whose centre lies on the boundary of a-shire and which does not contain any poles of $f(z)$. It will be shown that at most a finite number of $f(z), f'(z), \ldots$ have no zeros in C.

If not, there must be an infinite sequence of functions

$$\left\{\frac{f^{(m)}(z)}{m!}\right\}^{1/m} \qquad (m=\mu_1, \mu_2, \ldots) \tag{7·8}$$

which have no branch-points in C and so are regular in it.

A part C_1 of C lies in a-shire. Let z_0 be a point in this part. Then, by L_{301} (ii) and L_{303}, the determinations of (7·8) can be chosen in such a way that

$$\left\{\frac{f^{(m)}(z_0)}{m!}\right\}^{1/m} \to \frac{1}{z_0 - a}. \tag{7·9}$$

Now the limit function of (7·8) is regular in C_1, owing to the uniformity of the convergence, and its modulus is $|z-a|^{-1}$. It follows from these facts and from (7·9) that the limit function is $(z-a)^{-1}$ in C_1.

Again, C lies inside a region $[R, \delta]$, and so, by L_{301} (i), the sequence (7·8) is uniformly bounded in C. It follows from the Stieltjes-Vitali theorem,* in Stieltjes' original form, that (7·8) converges to $(z-a)^{-1}$ in the whole of C. But C contains part of the county of at least one other pole besides a, say that of a pole b, and, by L_{301} (ii),

$$\left|\frac{f^{(n)}(z)}{n!}\right|^{1/n} \to \frac{1}{|z-b|} \text{ in } b\text{-shire.}$$

We have now reached a contradiction, since $|z-a| > |z-b|$ in b-shire. This completes the proof of the theorem.

Pólya next defines the *boundary* (Grenzlage) of a sequence of plane sets $M_1, M_2, \ldots$ in the following manner. A neighbourhood of a point z being defined to be a circle of centre z, the points of the plane can be separated into three categories.

A point belongs to the first category if some neighbourhood of it contains only a finite number of points of $M_1, M_2, \ldots$.

A point belongs to the second category if every neighbourhood of it contains an infinite number of points of $M_1, M_2, \ldots$, but in some neighbourhood an infinite number of $M_1, M_2, \ldots$ have no points.

A point belongs to the third category if every neighbourhood of it contains points drawn from all the sets $M_1, M_2, \ldots$ except possibly a finite number.

The points of the third category form the *smallest boundary* of the sequence of sets $M_1, M_2, \ldots$, those of the second and third categories together the *largest boundary*. If these coincide, i.e. if

* Dienes (**1**), 156.

there are no points of the second category, the set is said to have a *boundary*.

Take M_n to be the set of zeros of $f^{(n)}(z)$. It has been seen that a point in a county belongs to the first category, and that all other points belong to the third category. Thus the aggregate of zeros has a definite boundary, namely the lines separating the counties.

We have considered the zeros of the functions $f(z), f'(z), \ldots$. It is by no means obvious that the α-points, i.e. the values of z for which $f(z), f'(z), \ldots$ take a given value α, have similar properties. This is however the case, the boundary of the aggregate being the same as that of the zeros. This follows from the fact that the sequence

$$|f'(z)-\alpha|, \quad \left|\frac{f''(z)-\alpha}{2!}\right|^{\frac{1}{2}} \ldots, \tag{7·10}$$

has the two properties of L_{301}; the sequence (7·10) is uniformly bounded in any $[R, \delta]$, and inside the counties we have

$$\lim \left|\frac{f^{(n)}(z)-\alpha}{n!}\right|^{1/n} = \lim \left|\frac{f^{(n)}(z)}{n!}\right|^{1/n}.$$

Theorem 6 shows that if $f(z)$ has only one pole E' is null, but it does not throw any light on the case when $f(z)$ has no poles, i.e. when $f(z)$ is an integral function. Pólya goes on to discuss the case of an integral function of the form

$$P(z)\, e^{Q(z)},$$

where $P(z)$, $Q(z)$ are polynomials, $Q(z)$ of degree $q \geqslant 2$, and shows that the limiting position of the aggregate of zeros consists of q concurrent semi-infinite lines; but even this case, simple as it looks, is very difficult to deal with, and it seems highly improbable that the set E' has a simple geometrical form in any wide class of cases.

§ 8. Theorems of Gontcharoff, Takenaka and Kakeya.

Gontcharoff's inequality. $G_n(z)$, in the series (7·1), is determined by the relations

$$G_n^{(m)}(a_m) = 0 \qquad (m = 0, 1, 2, \ldots, n-1),$$
$$G_n^{(n)}(z) = n!,$$

whence it is easy to see that

$$(8\cdot1)\qquad G_n(z)=n!\int_{a_0}^{z}dz'\int_{a_1}^{z'}dz''\dots\int_{a_{n-1}}^{z^{(n-1)}}dz^{(n)}$$
$$=n!\,G(z;a_0,a_1,\dots,a_{n-1})\qquad\text{(say)}.$$

Evidently

$$(8\cdot2)\quad G^{(m)}(z;a_0,\dots,a_{n-1})=G(z;a_m,a_{m+1},\dots,a_{n-1})\quad(0<m<n).$$

Again $f(z)=f(a_0)+\int_{a_0}^{z}f'(z')\,dz'$

$$=f(a_0)+\int_{a_0}^{z}dz'\left\{f'(a_1)+\int_{a_1}^{z'}f''(z'')\,dz''\right\}$$
$$=f(a_0)+G_1(z)f'(a_1)+\int_{a_0}^{z}dz'\int_{a_1}^{z'}f''(z'')\,dz'',$$

and so on, so that

$$(8\cdot3)\quad f(z)=f(a_0)+G_1(z)f'(a_1)+\dots$$
$$+\frac{G_{n-1}(z)}{(n-1)!}f^{(n-1)}(a_{n-1})+R_n(z),$$

where

$$(8\cdot4)\qquad R_n(z)=\int_{a_0}^{z}dz'\int_{a_1}^{z'}dz''\dots\int_{a_{n-1}}^{z^{(n-1)}}f^{(n)}(z^{(n)})\,dz^{(n)}.$$

Gontcharoff's (**1**) results are mostly founded on an inequality which he proved for integrals of this type. Write

$$I=\int_{a_0}^{z}dz'\dots\int_{a_{n-1}}^{z^{(n-1)}}F(z^{(n)})\,dz^{(n)},$$

and denote by $l_0, l_1, \dots, l_{n-1}$ the lines joining z to a_0, a_0 to a_1, …, a_{n-2} to a_{n-1}. Some of these lines may reduce to points. Let L denote the broken line formed by $l_0, l_1, \dots, l_{n-1}$, and let $F(z)$ be regular in a domain containing L, so that

$$|F(z)|\leqslant M \text{ on } L.$$

Write

$$t_i=|a_{n-1}-a_{n-2}|+\dots+|a_{i+1}-a_i|\qquad(i=0,1,\dots,n-2),$$
$$t_{n-1}=0,\quad t=t_0+|z-a_0|.$$

The paths of integration are all portions of L, in the direction of decreasing indices of the points a_i. Thus z' is a typical point of L between a_0 and z, z'' a point between a_1 and z', and so on. Denote by $t^{(k)}$ $(k=1, 2, \ldots, n)$ the length of the path $a_{n-1}z^{(k)}$ (along L), and let $t^{(k)}=t_{k-1}$ when $z^{(k)}=a_{k-1}$, and $t'=t$ when $z'=z$, so that

$$|I| \leqslant \int_{t_0}^{t} dt' \left| \int_{a_1}^{z'} dz'' \int_{a_2}^{z''} dz''' \ldots \int_{a_{n-1}}^{z^{(n-1)}} F(z^{(n)})\, dz^{(n)} \right|$$

$$\leqslant \int_{t_0}^{t} dt' \int_{t_1}^{t'} dt'' \left| \int_{a_2}^{z''} dz''' \ldots \int_{a_{n-1}}^{z^{(n-1)}} F(z^{(n)})\, dz^{(n)} \right|,$$

....

$$\leqslant M \int_{t_0}^{t} dt' \int_{t_1}^{t'} dt'' \ldots \int_{t_{n-1}}^{t^{(n-1)}} dt^{(n)}.$$

Now it is evident that

$$t_0 \leqslant t' \leqslant t, \quad t_1 \leqslant t'' \leqslant t', \quad \ldots, \quad t_{n-1} \leqslant t^{(n)} \leqslant t^{(n-1)},$$
$$t_{i+1} \leqslant t_i \qquad (0 \leqslant i \leqslant n-2),$$

so that

$$(8\cdot5) \qquad |I| \leqslant M \int_0^t dt' \int_0^{t'} dt'' \ldots \int_0^{t^{(n-1)}} dt^{(n)} = \frac{Mt^n}{n!}.$$

Write

$$u_n = |a_n - a_{n+1}| \qquad (n \geqslant 0),$$
$$s_0 = 0, \; s_n = \sum_{m=0}^{n-1} u_m \qquad (n \geqslant 1).$$

Then, taking $F(z) \equiv 1$, (8·5) gives

$$(8\cdot6) \quad |G(z; a_0, a_1, \ldots, a_{n-1})| \leqslant \frac{1}{n!}(|z-a_0| + s_{n-1})^n \qquad (n \geqslant 1).$$

Again, taking $F(z) = f^{(n)}(z)$,

$$(8\cdot7) \qquad |R_n(z)| \leqslant \frac{M_n}{n!}(|z-a_0| + s_{n-1})^n \qquad (n \geqslant 1),$$

where M_n denotes the maximum of $|f^{(n)}(z)|$ on L.

Gontcharoff's first theorem is as follows:

Theorem 7. *If*

(i) $\sum_{n=0}^{\infty} |a_n - a_{n+1}|$ *converges, so that* $a_n \to Z$;

(ii) $f(z)$ *is regular in the circle of centre* Z, *radius* R: *then the series*

$$(8\cdot8) \qquad G_0(z) f(a_0) + G_1(z) f'(a_1) + \frac{G_2(z)}{2!} f''(a_2) + \ldots$$

converges uniformly to $f(z)$ *in every circle interior to the circle of regularity.*

There is no loss of generality in taking $Z=0$.

It will first be shown that the series converges uniformly in

$$|z| \leqslant R' \qquad (0 < R' < R).$$

Let $R' < R'' < R$, and write

$$\rho_n = \sum_{m=n}^{\infty} |a_m - a_{m+1}|.$$

Then there is an integer N such that

$$\rho_N < \tfrac{1}{2}(R'' - R'). \tag{8·9}$$

Denote by $M(\phi, r)$ the maximum modulus in $|z| \leqslant r$ of a function $\phi(z)$, supposed regular in some larger circle. Then, as follows readily from Cauchy's integral,

$$\frac{M(\phi^{(n)}, r)}{n!} < \frac{M(\phi, R)}{(R-r)^n} \qquad (r < R). \tag{8·10}$$

Now, using (8·2), if $n > N$,

$$\frac{1}{n!} f^{(n)}(a_n)\, G_n^{(N)}(z) = f^{(n)}(a_n)\, G(z; a_N, a_{N+1}, \ldots, a_{n-1}),$$

so that, using (8·6) and (8·10),

$$\left|\frac{1}{n!} f^{(n)}(a_n)\, G_n^{(N)}(z)\right| \leqslant M(f^{(n)}, |a_n|)\, |G(z; a_N, \ldots, a_{n-1})|$$

$$< \frac{(n-N)!\, M(f^{(N)}, R'')\,(|z-a_N| + \ldots + |a_{n-2} - a_{n-1}|)^{n-N}}{(R'' - |a_n|)^{n-N}\,(n-N)!},$$

n being supposed so large that $|a_n| < R'$. Moreover,

$$|z - a_N| \leqslant |z| + |a_N| \leqslant R' + \rho_N,$$

and

$$|a_N - a_{N+1}| + \ldots + |a_{n-2} - a_{n-1}| \leqslant \rho_N.$$

Hence

$$\left|\frac{1}{n!} f^{(n)}(a_n)\, G_n^{(N)}(z)\right|^{1/n} \leqslant \{M(f^{(N)}, R'')\}^{1/n} \left\{\frac{R' + 2\rho_N}{R'' - |a_n|}\right\}^{1-N/n},$$

and, as $n \to \infty$, the limit of the expression on the right is

$$(R' + 2\rho_N)/R'',$$

which is less than unity. Thus the series obtained by differentiating (8·8) N times is uniformly convergent in $|z| \leqslant R'$, and the same is therefore true of (8·8) itself. The sum of this series is thus regular in $|z| < R$.

Again, consider the remainder after n terms of the series obtained by differentiating (8·8) N times, N being now chosen so that

$$\rho_N < \tfrac{1}{4}R' \qquad (0 < R' < R).$$

If $|z| \leqslant \frac{1}{4}R'$, (8·7) gives

$$\begin{aligned} |R_n^{(N)}(z)| &< \frac{1}{(n-N)!} M\left(f^{(n)}, \frac{R'}{4}\right) \qquad (|z-a_N| + \rho_N)^{n-N} \\ &< \frac{M(f^{(N)}, R')}{\left(R' - \frac{R'}{4}\right)^{n-N}} \qquad (|z| + 2\rho_N)^{n-N} \\ &\leqslant M(f^{(N)}, R') \left(\frac{\frac{1}{4}R' + 2\rho_N}{\frac{3}{4}R'}\right)^{n-N} \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

since $(\frac{1}{4}R' + 2\rho_N)/\frac{3}{4}R'$ is less than unity.

Hence $\quad |R_n(z)| \to 0 \quad \text{as} \quad n \to \infty \qquad (|z| \leqslant \frac{1}{4}R'),$

and so (8·8) has sum $f(z)$ in the circle $|z| = \frac{1}{4}R'$. As the sum of the series is a regular function it must be $f(z)$ in the whole circle $|z| < R$.

An obvious corollary of Theorem 7 is as follows:

L_{304}. *Let $f(z)$ $(\not\equiv 0)$ be regular in a domain D, and let $a_0, a_1, \ldots$ be zeros of $f(z), f'(z), \ldots$ respectively. Then $\Sigma |a_n - a_{n+1}|$ cannot converge if the sequence $a_0, a_1, \ldots$ has a limit Z inside D.*

In particular, if $f(z)$ is an integral function and a_n is any zero of $f^{(n)}(z)$, $\Sigma |a_n - a_{n+1}|$ diverges.

As an example of a case where Z is on the boundary of D and $\Sigma |a_n - a_{n+1}|$ converges, consider the function $e^{-1/z}$, for which

$$f^{(n)}(x) \to 0 \quad \text{as} \quad x \to 0$$

by real positive values. Moreover $f''(\frac{1}{2}) = 0$, so that, by Rolle's theorem, there is a sequence $a_2 > a_3 > \ldots > 0$ of zeros of $f''(z)$, $f'''(z), \ldots$, the limit of the sequence being $Z = 0$.

In Theorem 7 the condition that $\Sigma(a_n - a_{n+1})$ converges absolutely is all important. For consider the function $(1+z^2)^{-1}$. The "county boundary" is in this case the real axis, so, by what has been shown, if Z is any real number, a sequence $\{a_n\}$ of zeros can be found so that $\Sigma(a_n - a_{n+1})$ converges to Z. By L_{304}, the convergence can never be absolute, but it may be shown that the divergence of $\Sigma \mid a_n - a_{n+1} \mid$ can be as slow as we please.

It has been seen that a sequence of zeros cannot converge absolutely to a point at which the function is regular. Gontcharoff proves further that

$$(8{\cdot}11) \qquad \overline{\lim_{n\to\infty}}\, n \mid a_n - a_{n+1} \mid \geqslant \frac{R}{e},$$

a_n being a zero of $f^{(n)}(z)$ in $|z| < R$, and Kakeya (**1**), extending a result of Takenaka (**2, 3**), has shown that

$$(8{\cdot}12) \qquad \overline{\lim_{n\to\infty}}\, n \mid a_n \mid \geqslant R \log 2.$$

Consider once more the set E consisting of the zeros of a function and all its derivatives. Gontcharoff defines a point Z of E' to be *regular* if it is possible to assign a sequence of points $\{a_n\}$ such that $f^{(n)}(a_n) = 0$ and $a_n \to Z$. Pólya's investigation shows that a meromorphic function possesses only regular points. An integral function may, however, possess irregular points, e.g.

$$f(z) = \sin z, \quad Z = k\frac{\pi}{2} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Let Z be a regular point of a function $f(z)$, supposed analytic at Z. Let a_n be a zero of $f^{(n)}(z)$ such that the circle

$$|z - Z| < |a_n - Z|$$

does not contain any zeros ($a_n = Z$ if $f^{(n)}(Z) = 0$). Now define the *order* of Z to be α, where

$$(8{\cdot}13) \qquad \frac{1}{\alpha} = \frac{1}{R} \overline{\lim_{n\to\infty}}\, n \mid a_n - Z \mid,$$

R being the radius of convergence of the Taylor series of $f(z)$ at Z. Then (8·12) shows that α cannot exceed $1/\log 2$.

It would be interesting to know the "best possible" result in this direction. Gontcharoff conjectures that the order can never exceed $2/\pi$, the value attained in the case of the function

$$f(z) = \frac{1}{z^2 + a^2}.$$

Takenaka's theorem. Gontcharoff and Takenaka prove many theorems concerning integral functions, those of the former usually involving

$$s_n = \sum_{m=0}^{n-1} | a_m - a_{m+1} |,$$

and those of the latter $|a_n|$. From these we select the following, a particular case of one of Takenaka's theorems.* The proof given below is suggested by Kakeya's proof of (8·12).

Theorem 8. *If every derivative of an integral function $f(z)$ has a zero inside or on the unit circle and if*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < \log 2, \tag{8·14}$$

then $f(z)$ is a constant.

Let $G_0(z), G_1(z), \ldots$ be the basic set associated with a sequence $a_0, a_1, \ldots$ satisfying

$$|a_n| \leqslant k < \log 2 \qquad (n \geqslant 0), \tag{8·15}$$

so that

$$z^n = a_0^n G_0(z) + n a_1^{n-1} G_1(z) + \frac{n(n-1)}{2!} a_2^{n-2} G_2(z) + \ldots + G_n(z). \tag{8·16}$$

Given R, suppose that

$$|G_m(z)| \leqslant M m! \qquad (|z| \leqslant R,\ m = 0, 1, \ldots, n-1).$$

Then (8·16) gives, for $|z| \leqslant R$,

$$\begin{aligned} |G_n(z)| &\leqslant R^n + k^n M + n k^{n-1} M 1! + \frac{n(n-1)}{2!} k^{n-2} M 2! + \ldots \\ &= R^n + M n! \left(k + \frac{k^2}{2!} + \frac{k^3}{3!} + \ldots + \frac{k^n}{n!} \right) \\ &< \left\{ \frac{R^n}{n!} + M(e^k - 1) \right\} n! \\ &\leqslant M n!, \end{aligned}$$

provided that

$$\frac{R^n}{n!} \leqslant (2 - e^k) M.$$

* Takenaka (4).

Choose $M \geqslant 1$ so that this is satisfied for all n. Then, by induction,

$$|G_n(z)| \leqslant Mn! \qquad (|z| \leqslant R,\ n \geqslant 0).$$

Again, by (8·16),

$$\omega_n(R) \leqslant M\{k^n + nk^{n-1}1! + \ldots + n!\} < Mn!e^k.$$

It follows from (3·7), ..., (3·10) that the increase of the set of polynomials does not exceed order 1, type 1.

(8·14) implies that $f(z)$ is of increase not greater than order 1, type $l < \log 2$. Let $l < k < \log 2$. Then the increase of

$$F(z) = f\left(\frac{z}{k}\right)$$

does not exceed order 1, type $l/k < 1$, and so, by L_{105},

$$F(z) = \sum_{n=0}^{\infty} \frac{G_n(z)}{n!} F^{(n)}(a_n),$$

or

$$f\left(\frac{z}{k}\right) = \sum_{0}^{\infty} \frac{G_n(z)}{n!} \frac{1}{k^n} f^{(n)}\left(\frac{a_n}{k}\right). \tag{8·17}$$

If each of $f'(z)$, $f''(z)$, ... vanishes in $|z| \leqslant 1$, we may take the zeros to be the points a_1/k, a_2/k, ..., in virtue of (8·15), and with $a_0 = 0$, (8·17) gives

$$f\left(\frac{z}{k}\right) = G_0(z) f(0) = f(0).$$

The condition (8·14) is probably not "best possible". Consideration of the function

$$\sin\frac{\pi}{4}z - \cos\frac{\pi}{4}z$$

shows that $\log 2$ cannot be replaced by any number larger than $\pi/4$, and this may well be the true value.

§ 9. The two-point boundary problem.

Mention has been made of Lidstone's series, the basic series associated with the operators $f(1)$, $f(0)$, $f''(1)$, $f''(0)$, ... It can be shown that it represents all integral functions such that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} < \pi. \tag{9·1}$$

Thus any integral function which satisfies (9·1) is determined when

(9·2) $f(1),\ f''(1),\ f^{(\mathrm{iv})}(1), \ldots;\ f(0),\ f''(0),\ f^{(\mathrm{iv})}(0), \ldots$

are given, just as any function regular at the origin is determined when $f(0),\ f'(0),\ f''(0), \ldots$ are given. This remark suggests a general problem. Suppose that we are given two increasing sequences of integers, $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$. What conditions must they satisfy in order that a knowledge of

(9·3) $f^{(p_1)}(1),\ f^{(p_2)}(1), \ldots;\ f^{(q_1)}(0),\ f^{(q_2)}(0), \ldots$

may determine all integral functions of sufficiently slow growth? At the moment we are not concerned with the class of functions to be determined, i.e. with the analogue of (9·1). All we want to secure is that there shall be, so to speak, just enough derivatives assigned. In other words, that the operators (9·3) shall form a basic set.

Now it is clear that the matrix of these operators is row-finite and it will have a unique row-finite reciprocal if and only if there is a unique set of polynomials $\pi_n(z)$, $\zeta_n(z)$ such that

$$(9\cdot4)\qquad \begin{cases} \pi_n^{(t)}(1) = 1, & t = p_n; \\ \qquad\quad\ = 0, & t = p_r \quad (r \neq n); \\ \pi_n^{(t)}(0) = 0, & t = q_r \quad (r \geqslant 1). \end{cases}$$

$$(9\cdot5)\qquad \begin{cases} \zeta_n^{(t)}(0) = 1, & t = q_n; \\ \qquad\quad\ = 0, & t = q_r \quad (r \neq n); \\ \zeta_n^{(t)}(1) = 0, & t = p_r \quad (r \geqslant 1). \end{cases}$$

A set of polynomials satisfying these equations will be called a *standard set* in relation to the pair of sequences $(p_1, p_2, \ldots; q_1, q_2, \ldots)$, which will be written briefly $(p; q)$. If (9·3) is basic so that there is a unique standard set, the pair $(p; q)$ will be said to be *complete*. If more than one standard set exists, $(p; q)$ will be called *indeterminate* and, if no standard set exists, *redundant*.

The three cases can be distinguished by means of the function $D(m)$ defined as follows:

(9·6) $D(m) =$ number of p's and q's which are less than m.

Theorem 9. *In order that a pair $(p; q)$ may be complete it is necessary and sufficient that*

$$D(m) \geqslant m \qquad (m \geqslant 1), \tag{9·7}$$

and

$$D(m_r) = m_r, \textit{ for an infinite sequence } m_1, m_2, \ldots. \tag{9·8}$$

If one or more p's and q's are removed from a complete pair it becomes indeterminate; and, conversely, an indeterminate pair can be made complete by adding p's and q's.

We need an algebraic theorem due to Aitken and Zia-uddin.*

$\mathbf{L}_{305}$. *If $a, b, \ldots, a', b', \ldots$ are positive integers satisfying the inequalities*

$$a < b < \ldots < k, \quad a' < b' < \ldots < k',$$
$$a' \leqslant a, \quad b' \leqslant b, \quad \ldots, \quad k' \leqslant k,$$

then

$$\begin{vmatrix} \{a, a'\} & \{b, a'\} & \ldots & \{k, a'\} \\ \{a, b'\} & \{b, b'\} & \ldots & \{k, b'\} \\ \ldots & \ldots & \ldots & \ldots \\ \{a, k'\} & \{b, k'\} & \ldots & \{k, k'\} \end{vmatrix} \geqslant 1,$$

where $\qquad \{a, a'\} = a(a-1)(a-2)\ldots(a-a'+1), \ldots,$

and the determinant may be of any order.

To prove Theorem 9 (**5**), assume in the first place that (9·7), (9·8) are satisfied. Let $v_1, v_2, \ldots$ be the sequence complementary to $q_1, q_2, \ldots$ with respect to $0, 1, 2, \ldots$, and let

$$V(m) = \text{number of } v\text{'s which are less than } m.$$

Similarly define $P(m)$, $Q(m)$. Then

$$V(m) = m - Q(m)$$

and (9·7) can be written

$$P(m) \geqslant V(m) \qquad (m \geqslant 1),$$

or

$$p_r \leqslant v_r \qquad (r \geqslant 1). \tag{9·9}$$

To determine $\pi_n(z)$ for some fixed value of n, choose N to be any one of the integers $m_1, m_2, \ldots$ which is such that $V(N) \geqslant n$,

* Zia-uddin (**1**).

and assume that $\pi_n(z)$ is of degree less than N. If we write $V(N) = V$ it is clear that $\pi_n(z)$ must be of the form

$$d_1 z^{v_1} + d_2 z^{v_2} + \dots + d_V z^{v_V}.$$

For the last condition of (9·4) requires that only v's can occur as indices and v_V is the largest v less than N. The remaining conditions of (9·4) furnish V equations to determine the coefficients $d_1, d_2, \dots$, namely

$$(9{\cdot}10)\qquad \begin{cases} d_1\{v_1, p_1\} + d_2\{v_2, p_1\} + \dots + d_V\{v_V, p_1\} = 0, \\ d_1\{v_1, p_2\} + \dots \qquad\qquad\qquad + d_V\{v_V, p_2\} = 0, \\ \quad\dots \qquad\qquad \dots \qquad\qquad \dots \qquad \dots \\ d_1\{v_1, p_n\} + \dots \qquad\qquad\qquad + d_V\{v_V, p_n\} = 1, \\ \quad\dots \qquad\qquad \dots \qquad\qquad \dots \qquad \dots \\ d_1\{v_1, p_V\} + \dots \qquad\qquad\qquad + d_V\{v_V, p_V\} = 0. \end{cases}$$

The inequalities of L_{305} being satisfied in virtue of (9·9), these equations have a non-zero determinant and so a unique solution. Since N may be arbitrarily large, it follows that there is one and only one polynomial $\pi_n(z)$ satisfying (9·4). The existence of a unique set of polynomials $\zeta_n(z)$ is established in the same way, and the pair $(p; q)$ is therefore complete.

Next suppose that (9·7) is satisfied but that (9·8) is not, so that

$$(9{\cdot}11)\qquad D(m) > m \qquad (m \geqslant M).$$

M may be taken to be the smallest integer for which this is true. Thus

$$D(M-1) = M - 1,$$
$$D(M) > M.$$

$M-1$ is therefore both a p and a q. Omit $M-1$, considered as a q, from the pair of sequences. Then we are left with a new pair for which (9·7) is satisfied. If this pair does not satisfy (9·8), (9·11) must be true with $M' > M$ in place of M. We can now omit another q, and so on. In this way a pair $(p; w)$ is obtained, $w_1, w_2, \dots$ being a subsequence of $q_1, q_2, \dots$ for which (9·7), (9·8) are satisfied. This pair is therefore complete.

Let q_n be one of the omitted q's. If a standard set of polynomials in relation to the pair $(p; q)$ exists, the polynomial $\zeta_n(z)$ is such that

$$\zeta_n^{(t)}(0)=0, \qquad t=w_1, w_2, \ldots,$$
$$\zeta_n^{(t)}(1)=0, \qquad t=p_1, p_2, \ldots.$$

Constant multiples of $\zeta_n(z)$ can therefore be added to the polynomials of the standard set attached to the pair $(p; w)$ without destroying their properties; and, since $\zeta_n(z)$ is not identically zero, the pair $(p; w)$ cannot be complete. The contradiction implies that $(p; q)$ is redundant.

If (9·7) is not satisfied, so that

$$D(M)<M$$

for some M, it is clear that a non-zero polynomial $\rho(z)$ of degree $M-1$ can be determined so that

$$\rho^{(t)}(1)=0, \qquad t=p_1, p_2, \ldots,$$
$$\rho^{(t)}(0)=0, \qquad t=q_1, q_2, \ldots.$$

For these are $P(M)+Q(M)=D(M)<M$ equations to determine M coefficients. If a standard set of polynomials exists, constant multiples of $\rho(z)$ can be added to them without destroying their properties. Thus the pair cannot be complete.

Next suppose that some p's and q's are removed from a complete pair $(p; q)$. Let $F(z)$ be the polynomial corresponding to an omitted integer. Then $\lambda F(z)$, where λ is any constant, can be added to the surviving polynomials without destroying their property of being standard with respect to the surviving p's and q's. Hence the new pair of sequences is indeterminate.

Finally suppose that $(p; q)$ is an indeterminate pair. It has been seen that in this case (9·7) cannot be satisfied. Then let M be the first integer such that

$$D(M)<M.$$

Augment $(p; q)$ by adding to it, in order, firstly to the p-sequence and secondly to the q-sequence, such of 0, 1, 2, ... as are not already in them, until a pair is constructed for which

$$D_1(M)=M.$$

If (9·7) is still not satisfied repeat the process, and so on. The resulting pair $(p'; q')$ satisfies (9·7).

If $(p'; q')$ is not complete there is an integer L such that

$$(9{\cdot}12)\qquad \begin{cases} D'(m) > m & (m \geqslant L), \\ D'(L-1) = L-1. \end{cases}$$

This being so it is clear that all p's and q's which are greater than $L-2$ belong to the original sequence $(p; q)$, e.g. if

$$(p; q) = (2, 3, \ldots; 2), \quad \text{then} \quad (p'; q') = (0, 2, 3, \ldots; 0, 2),$$

and these are identical for $p, q > 1$.

Let $\{\pi_n(z), \zeta_n(z)\}$ be a standard set associated with $(p; q)$. Then $\{\pi_n^{(L-1)}(z), \zeta_n^{(L-1)}(z)\}$ is a standard set associated with the pair formed by removing from $(p; q)$ those members which are less than $L-1$, and subtracting $L-1$ from the survivors; and this is the same as the pair $(p''; q'')$ formed by applying the same process to $(p'; q')$. Now $(p''; q'')$ satisfies the condition

$$(9{\cdot}13)\qquad D''(m) > m \qquad (m \geqslant 1),$$

since, using (9·12),

$$D''(m-L+1) = D'(m) - D'(L-1) \qquad (m \geqslant L)$$
$$> m - L + 1.$$

It has been seen that (9·13) implies that $(p''; q'')$ is redundant, and this is impossible since the pair has a standard set associated with it. The contradiction implies that $(p'; q')$ is complete.

Theorem 9 has an interesting corollary.

$\mathbf{L}_{306}$. *A set of operators* $f^{(p_1)}(1), f^{(p_2)}(1), \ldots; f^{(q_1)}(0), \ldots$ *cannot be basic unless it is regular.*

If $(p; q)$ is a complete pair it follows from L_{103} that the series

$$\pi_1(z) f^{(p_1)}(1) + \ldots + \zeta_1(z) f^{(q_1)}(0) + \ldots$$

represents all integral functions of sufficiently slow growth, i.e. all which satisfy a condition

$$(9{\cdot}14)\qquad |a_n| < \phi(n).$$

Moreover, if A_n, B_n are sufficiently small there is a unique function of this class such that

$$(9{\cdot}15)\qquad f^{(p_n)}(1) = A_n, \quad f^{(q_n)}(0) = B_n \qquad (n \geqslant 1),$$

namely the function

$$(9{\cdot}16)\qquad A_1 \pi_1(z) + \ldots + B_1 \zeta_1(z) + \ldots.$$

For evidently there exists a positive function $\psi(n)$ which is such that, if

$$(9\cdot17) \qquad |A_n| + |B_n| < \psi(n),$$

then the function (9·16) satisfies (9·14).

If $(p; q)$ is an indeterminate pair we have seen that it can be augmented into a complete pair $(p'; q')$, so that, if

$$|A_n| + |B_n| < \psi'(n),$$

more than one solution of (9·15) exists. Lastly, if $(p; q)$ is redundant, there exists a set of values of A_n, B_n, tending to zero as rapidly as we please, such that no function satisfying (9·14) and (9·15) exists. Thus it may be said that a complete pair contains just enough conditions to determine a function of sufficiently slow growth, an indeterminate pair too few, and a redundant pair too many.

We may ask a further question. Suppose that $f(z)$, $g(z)$ are any integral functions and $(p; q)$ any complete pair. Is it always possible to find an integral function $h(z)$ such that

$$(9\cdot18) \qquad h^{(p_n)}(1) = f^{(p_n)}(1), \quad h^{(q_n)}(0) = g^{(q_n)}(0) \qquad (n \geqslant 1),$$

and, if so, what can be said about the order of $h(z)$? In the case when $(p; q)$ is the Lidstone pair $(0, 2, 4, \ldots; 0, 2, 4, \ldots)$ the question can be answered without much difficulty by making use of Theorem 3. $h(z)$ exists and can be chosen of order not exceeding the greater of the orders of $f(z)$, $g(z)$. Indeed this result is equivalent to Theorem 3—either can be deduced from the other by elementary reasoning. The case of an arbitrary complete pair seems to be very difficult, and no progress has been made.

CHAPTER IV

INTERPOLATION AT THE INTEGERS

§ 10. The Gregory-Newton series.

The earliest basic series was discovered by James Gregory in 1670. This is the series*

$$(10\cdot1)\quad f(0)+z\Delta f(0)+\frac{z(z-1)}{2!}\Delta^2 f(0)+\frac{z(z-1)(z-2)}{3!}\Delta^3 f(0)+\dots,$$

where

$$(10\cdot2)\quad \Delta f(0)=f(1)-f(0),$$

$$\Delta^2 f(0)=\Delta f(1)-\Delta f(0)=f(2)-2f(1)+f(0),\dots$$

which solves the fundamental problem of interpolation, that of finding a polynomial of degree n (or less) which takes given values $f(0), f(1), \dots, f(n)$ at $z=0, 1, \dots, n$. The series has been extensively studied† and has important applications.

Theorem 10. (i) *Let $f(z)$ be an integral function which satisfies the condition*

$$(10\cdot3)\qquad \varlimsup_{r\to\infty}\frac{\log M(r)}{r}<\log 2.$$

Then the series (10·1) *converges to $f(z)$ uniformly in any finite region of the plane.*

(ii) *Let $d_0, d_1, \dots$ be a sequence such that*

$$(10\cdot4)\qquad \varlimsup_{n\to\infty}|d_n|^{1/n}=k<1.$$

Then the series

$$(10\cdot5)\qquad d_0+zd_1+\frac{z(z-1)}{2!}d_2+\dots$$

* See Whittaker and Robinson (**1**), Chapter I.

† The account given below is mainly based on Landau (**1**) and Okada (**1**). Nörlund (**1, 2**) gives a much longer and more detailed discussion.

converges uniformly in any finite region of the plane to an integral function $g(z)$ *which satisfies the condition*

$$\varlimsup_{r\to\infty}\frac{\log M(r)}{r}\leqslant\log\frac{1}{1-k}. \tag{10·6}$$

It is readily proved by induction that

$$\frac{1}{t-z}=\frac{1}{t}+\frac{z}{t(t-1)}+\frac{z(z-1)}{t(t-1)(t-2)}+\ldots+\frac{z(z-1)\ldots(z-n+1)}{t(t-1)\ldots(t-n)}+\frac{z(z-1)\ldots(z-n)}{t(t-1)\ldots(t-n)(t-z)},$$

so that $f(z)=\dfrac{1}{2\pi i}\displaystyle\int_{C_n}\frac{f(t)}{t-z}\,dt$

$$=f(0)+z\Delta f(0)+\ldots+\frac{z(z-1)\ldots(z-n+1)}{n!}\Delta^n f(0)+\frac{1}{2\pi i}\int_{C_n}\frac{z(z-1)\ldots(z-n)}{t(t-1)\ldots(t-n)}\frac{f(t)}{t-z}\,dt, \tag{10·7}$$

where C_n may be taken to be the circle $|t|=2n$, supposed large enough to enclose the given region of the z-plane.

Now, writing $z(z-1)\ldots(z-n)=(-)^n n!\,u_n(z)$, we have

$$\frac{u_n(z)}{u_{n-1}(z)}=1-\frac{z}{n},$$

so that
$$u_n(z)\sim ln^{-z}\qquad (l\neq 0),$$
and
$$|u_n(z)|<Kn^{-x}\qquad (z=x+iy).$$

Thus the remainder term in (10·7), R_n say, satisfies the inequality

$$|R_n|<Kn^{-x}n!\int_{C_n}\left|\frac{f(t)\,dt}{t(t-1)\ldots(t-n)(t-z)}\right|$$

$$<\frac{Kn^{-x}}{2n-|z|}P_n, \tag{10·8}$$

where
$$P_n=n!\int_{-\pi}^{\pi}\left|\frac{f(t)}{(t-1)\ldots(t-n)}\right|d\theta\qquad (t=2ne^{i\theta})$$

$$<Kn!\,e^{2n\lambda}\int_{-\pi}^{\pi}\left|\frac{1}{(t-1)\ldots(t-n)}\right|d\theta\qquad (\lambda<\log 2)$$

$$\leqslant Kn!\,e^{2n\lambda}\int_{-\pi}^{\pi}\frac{d\theta}{\prod\limits_{s=1}^{n}\{2n-s\cos\theta\}}<Ke^{2n(\lambda-\log 2)}, \tag{10·9}$$

since

$$|t-s|=\sqrt{(4n^2-4ns\cos\theta+s^2)}\geqslant 2n-s\cos\theta \qquad (1\leqslant s\leqslant n),$$

and
$$n!\,2^{2n}\int_{-\pi}^{\pi}\frac{d\theta}{\prod_{s=1}^{n}\{2n-s\cos\theta\}}<A.$$

The last inequality is proved as follows. We have

$$n!\,2^{2n}\frac{1}{\prod_{s=1}^{n}(2n-s)}=n!\,2^{2n}\frac{(n-1)!}{(2n-1)!}=2\,.\,2^{2n}\frac{n!\,n!}{(2n)!}<A\sqrt{n},$$

so it is enough to show that

$$\int_{-\pi}^{\pi}\psi(\theta,n)\,d\theta<\frac{A}{\sqrt{n}}\quad\text{where } \psi(\theta,n)=\prod_{s=1}^{n}\left\{\frac{2n-s}{2n-s\cos\theta}\right\}.$$

Now $\quad 1-\cos\theta=\dfrac{\theta^2}{2}-\dfrac{\theta^4}{24}+\ldots\geqslant\dfrac{\theta^2}{2}-\dfrac{\theta^4}{24}\geqslant\dfrac{\theta^2}{2}-\dfrac{10\theta^2}{24}=\dfrac{\theta^2}{12},$

for $-\pi\leqslant\theta\leqslant\pi$, and

$$\frac{1}{1+y}\leqslant e^{-y+\frac{1}{2}y^2}\leqslant e^{-y+\frac{1}{2}y}=e^{-\frac{1}{2}y} \qquad (0\leqslant y\leqslant 1),$$

so that, for $\qquad 0\leqslant\eta\leqslant\frac{1}{2},\quad -\pi\leqslant\theta\leqslant\pi,$

$$\frac{1-\eta}{1-\eta\cos\theta}=\frac{1}{1+\dfrac{\eta}{1-\eta}(1-\cos\theta)}\leqslant\frac{1}{1+\eta(1-\cos\theta)}$$
$$\leqslant e^{-\frac{1}{2}\eta(1-\cos\theta)}\leqslant e^{-\frac{1}{24}\eta\theta^2}.$$

Hence
$$\psi(\theta,n)=\prod_{s=1}^{n}\left\{\frac{1-\dfrac{s}{2n}}{1-\dfrac{s}{2n}\cos\theta}\right\}\leqslant e^{-\frac{\theta^2}{48n}\sum_{s=1}^{n}s}$$
$$=e^{-\frac{1}{96}(n+1)\theta^2}\leqslant e^{-\frac{n}{96}\theta^2} \qquad (-\pi\leqslant\theta\leqslant\pi,\ n=1,2,\ldots),$$

and
$$\int_{-\pi}^{\pi}\psi(\theta,n)\,d\theta\leqslant\int_{-\infty}^{\infty}e^{-\frac{n}{96}\theta^2}\,d\theta<\frac{A}{\sqrt{n}}.$$

The first part of Theorem 10 follows from (10·8), (10·9). To prove the second part, take $l>k$ so that

$$|d_n|\leqslant l^n \qquad (n>N).$$

Then

$$|g(re^{i\theta})| \leqslant |d_0| + r|d_1| + \frac{r(r+1)}{2!}|d_2| + \ldots$$

$$+ \frac{r(r+1)\ldots(r+N-1)}{N!}|d_N| + \frac{r(r+1)\ldots(r+N)}{(N+1)!}l^{N+1}$$

$$+ \frac{r(r+1)\ldots(r+N+1)}{(N+2)!}l^{N+2} + \ldots < P(r) + \left(\frac{1}{1-l}\right)^r,$$

where $P(r)$ is a polynomial of degree N. Hence

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} \leqslant \log \frac{1}{1-l},$$

and (10·6) follows.

A corollary of Theorem 10 is the beautiful theorem of Pólya and Hardy,* "2^z is the smallest transcendental integral function which takes integral values at $z=0, 1, 2, \ldots$". More precisely,

Theorem 11. *If $f(z)$ is an integral function which satisfies* (10·3) *and if $f(0), f(1), \ldots$ are integers, then $f(z)$ is a polynomial.*

For, by Theorem 10,

$$f(-1) = f(0) - \Delta f(0) + \Delta^2 f(0) - \ldots,$$

and, as this series is convergent,

$$|\Delta^n f(0)| < 1 \qquad (n > N).$$

But, by (10·2), $\Delta f(0), \Delta^2 f(0), \ldots$ are all integers (or zero). Hence

$$\Delta^n f(0) = 0 \qquad (n > N),$$

and so (10·1), which represents $f(z)$, reduces to a polynomial of degree $N-1$.

Singularities of Taylor series. Some important theorems relating the singularities of a Taylor series on its circle of convergence to the size and density of the coefficients depend on interpolatory principles. We prove the simplest of these, commending to the reader a long memoir of Pólya (**5**) and Bernstein's tract (**1**) for a fuller discussion and for further references.

* Pólya (**1**) proved the theorem with an additional factor $\sqrt{r}$ in (10·3). This was removed by Hardy (**1**). Pólya (**3**) has since proved a more precise result, (10·3) being replaced by $\overline{\lim}\, 2^{-r} M(r) < 1$.

Theorem 12. *Let* $f(z)=\Sigma a_n z^n$ *be regular in* $|z|<1$ *and let* $z=1$ *be the only singularity on* $|z|=1$ *and let it be an isolated non-critical singularity.** *Then*

(10·10) $$a_n = g(n) + b_n,$$

where $g(z)$ *is an integral function satisfying the condition*

(10·11) $$\frac{\log M(r)}{r} \to 0,$$

and

(10·12) $$\overline{\lim_{n\to\infty}} \, |b_n|^{1/n} < 1.$$

This result is due to Wigert† (**1**). To prove it, consider the Laurent expansion

$$f(z)=\sum_{n=1}^{\infty} \frac{A_n}{(z-1)^n} + \sum_{n=0}^{\infty} B_n (z-1)^n = F(z)+G(z) \quad \text{(say)}.$$

This converges at all points near $z=1$, so $\Sigma A_n w^n$ is an integral function of w and $F(z)=\Sigma c_n z^n$ has no singularities except $z=1$. $G(z)=f(z)-F(z)$ has therefore no singularities except those of $f(z)$ other than $z=1$, i.e. it is regular in $|z|<R$, where $R>1$. Hence $G(z)=\Sigma b_n z^n$, where b_n satisfies (10·12).

Again, it has been seen that $F(z)$ is an integral function of $(1-z)^{-1}$ and so of $z(1-z)^{-1}=(1-z)^{-1}-1$. Hence

$$(1-z)F(z)=\sum_{n=0}^{\infty} d_n \left(\frac{z}{1-z}\right)^n \qquad (|z|<1, \ |d_n|^{1/n}\to 0),$$

or
$$\sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} d_n z^n (1-z)^{-n-1},$$

and, on expanding the terms on the right and equating coefficients,

(10·13) $$c_n = d_0 + n d_1 + \frac{n(n-1)}{2!} d_2 + \ldots + d_n \qquad (n=0,1,2,\ldots).$$

By the second part of Theorem 10,

$$d_0 + z d_1 + \frac{z(z-1)}{2!} d_2 + \ldots$$

* I.e. there is a circle, centre $z=1$, in which $f(z)$ is uniform and at each point of which (except $z=1$) $f(z)$ is regular.

† For related results due to Faber, Carlson, Leau, etc., see Dienes (**1**), 337 ff.

converges to an integral function $g(z)$ which satisfies (10·11), and (10·13) is $c_n = g(n)$.

The singularities are related to the density of the coefficients by means of a theorem stated and used for this purpose by Faber (**1**), to the effect that if $f(z)$ is an integral function satisfying (10·11) and if $\alpha > 0$, then

$$(10\cdot 14) \qquad |f(n)| \geqslant e^{-\alpha n}$$

for almost all positive integers n, i.e. $N(r)$, the number of integers not greater than r for which (10·14) is false, is such that

$$(10\cdot 15) \qquad \frac{N(r)}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

It follows from this result that the density of the coefficients a_n in Theorem 12 is unity, i.e. that almost all of them are different from zero. For by (10·10), (10·12), if $a_n = 0$,

$$|g(n)| = |b_n| < e^{-\alpha n} \qquad (n > n_\alpha),$$

and this is false for almost all integers n.

The theorem of Faber-Pólya. Faber's proof that (10·14) is true for almost all n was not complete. The defect was remedied, many years later, by Pólya (**1**, Part II, Satz V). The proof given below is founded on Pólya's but seems better adapted for dealing with similar theorems, e.g. the corresponding result for functions of order two. It depends on an inequality for functions regular in the unit circle.

L$_{401}$. *Let*

(i) $f(z)$ *be regular in* $|z| < 1$;

(ii) $|f(z)| \leqslant M \qquad (|z| < 1)$;

(iii) $|f(a_k)| \leqslant L \qquad (k = 0, 1, \ldots, N)$,

$a_0, a_1, \ldots, a_N$ *being any distinct points in* $|z| \leqslant \tau < 1$. *Then*

$$(10\cdot 16) \quad |f(z)| \leqslant M\left(\frac{2\tau}{1+\tau^2}\right)^{N+1} + \frac{6}{1-\tau^4} L\Delta (2\tau)^N \qquad (|z| \leqslant \tau),$$

where

$$(10\cdot 17) \qquad \Delta = \sum_{k=0}^{N} \prod_i{}' \frac{1}{|a_i - a_k|}.$$

$\prod_i'$ means that $i=k$ is to be omitted. The proof depends on approximating to $f(z)$ by interpolation at the points a_i. Write

$$G(z)=\prod_{k=0}^{N}\frac{1-\bar{a}_k z}{z-a_k},$$

so that

$$H(z)=f(z)\,G(z)-\sum_{k=0}^{N}\frac{f(a_k)}{z-a_k}(1-\bar{a}_k a_k)\prod_i{}'\frac{1-\bar{a}_i a_k}{a_k-a_i}$$

is regular in $|z|<1$. Note for future reference that

$$|G(z)|=1 \qquad (|z|=1),$$

and
$$\left|\frac{u-v}{1-\bar{u}v}\right|\leqslant\frac{|u|+|v|}{1+|u|\,|v|} \qquad (|u|<1,\ |v|<1).$$

Evidently

$$\left|\prod_i{}'\frac{1-\bar{a}_i a_k}{a_k-a_i}\right|\leqslant\prod_i{}'\frac{1+\tau^2}{|a_k-a_i|}=(1+\tau^2)^N\,\Pi_k \quad \text{(say)}.$$

Hence, on $|z|=1$,

$$\left|\sum_{k=0}^{N}\frac{f(a_k)}{z-a_k}(1-\bar{a}_k a_k)\prod_i{}'\frac{1-\bar{a}_i a_k}{a_k-a_i}\right|\leqslant\frac{L}{1-\tau}(1+\tau^2)^N\sum_{k=0}^{N}\Pi_k,$$

using the fact that $\quad |1-\bar{a}_k a_k|=1-|a_k|^2\leqslant 1.$

The maximum modulus principle now gives

$$|H(z)|\leqslant M+\frac{L}{1-\tau}(1+\tau^2)^N\Delta \qquad (|z|<1),$$

and so, for $|z|\leqslant\tau$,

$$\begin{aligned}|f(z)|&\leqslant\frac{1}{|G(z)|}\left\{M+\frac{L}{1-\tau}(1+\tau^2)^N\Delta\right\}\\&\quad+\left|\sum_{k=0}^{N}\frac{f(a_k)}{1-\bar{a}_k z}(1-\bar{a}_k a_k)\prod_i{}'\frac{z-a_i}{1-\bar{a}_i z}\prod_i{}'\frac{1-\bar{a}_i a_k}{a_k-a_i}\right|\\&\leqslant\left(\frac{2\tau}{1+\tau^2}\right)^{N+1}\left\{M+\frac{L}{1-\tau}(1+\tau^2)^N\Delta\right\}\\&\quad+\frac{L}{1-\tau^2}\left(\frac{2\tau}{1+\tau^2}\right)^N(1+\tau^2)^N\Delta\\&=M\left(\frac{2\tau}{1+\tau^2}\right)^{N+1}+L\Delta\,(2\tau)^N\frac{(1+2\tau+3\tau^2)}{1-\tau^4},\end{aligned}$$

which gives (10·16).

The Faber-Pólya theorem will be proved in the following more general form.

Theorem 13. *Let $f(z)$ be an integral function such that*

$$\frac{\log M(r)}{r} \to 0, \tag{10·18}$$

and let α, λ be given positive numbers. Then

$$|f(z)| \geqslant e^{-\alpha n} \qquad (\lambda n \leqslant |z| \leqslant \lambda(n+1)), \tag{10·19}$$

for almost every integer n.

Denoting by $I(r)$ the number of integers not greater than r for which (10·19) is false, the result is equivalent to

$$\frac{I(r)}{r} \to 0. \tag{10·20}$$

Let $b_0, b_1, \ldots$ be points in $(0 \leqslant |z| \leqslant \lambda)$, $(\lambda \leqslant |z| \leqslant 2\lambda)$, ... respectively at which $|f(z)|$ attains its minimum value. Evidently

$$|b_{2n}| - |b_{2k}| \geqslant \lambda(n-k) \qquad (k=0, 1, \ldots, n-1),$$
$$|b_{2k}| - |b_{2n}| \geqslant \lambda(k-n) \qquad (k=n+1, n+2, \ldots).$$

Let R be a "large" positive number, γ a "small" one, and $c_0, c_1, \ldots, c_{\nu(R)}$ the b's of even suffix in

$$\gamma R \leqslant |z| \leqslant \tfrac{1}{2}R$$

for which

$$|f(b_n)| < e^{-\alpha n}. \tag{10·21}$$

Write $\qquad g(z) = f(Rz), \quad a_n = \dfrac{c_n}{R} \qquad (n=0, 1, \ldots, \nu),$

and apply (10·16) to the first $N+1$ (N will be chosen later) of these a's. Evidently

$$|a_n - a_k| \geqslant |a_n| - |a_k| = \frac{1}{R}(|c_n| - |c_k|)$$
$$\geqslant \frac{\lambda}{R}(n-k) \qquad (k=0, 1, \ldots, n-1),$$
$$|a_k - a_n| \geqslant \frac{\lambda}{R}(k-n) \qquad (k=n+1, n+2, \ldots),$$

so that

$$\Delta \leqslant \left(\frac{R}{\lambda}\right)^N \sum_{k=0}^{N} \frac{1}{k(k-1)\ldots 2 . 1 . 1 . 2 . \ldots (N-k)}$$
$$= \left(\frac{R}{\lambda}\right)^N \frac{2^N}{N!} < \left(\frac{R2e}{\lambda N}\right)^N,$$

and (10·16) gives (with $\tau = \frac{1}{2}$)

$$(10\cdot 22) \qquad |g(z)| \leqslant M(R)\left(\frac{4}{5}\right)^{N+1} + 8e^{-\alpha\gamma R}\left(\frac{R2e}{\lambda N}\right)^{N} \qquad (|z| \leqslant \tfrac{1}{2}).$$

Let β satisfy the inequalities

$$\beta \log \frac{1}{\beta} + \beta \left| \log \frac{2e}{\lambda} \right| < \tfrac{1}{2}\alpha\gamma, \qquad 0 < \beta \leqslant \frac{1}{e}.$$

Evidently such a number exists and

$$(10\cdot 23) \qquad t \log \frac{1}{t} + t \left| \log \frac{2e}{\lambda} \right| < \tfrac{1}{2}\alpha\gamma \qquad (0 < t \leqslant \beta).$$

Now $I(r) = I_1(r) + I_2(r)$, where $I_1(r)$ is the number of b's of even suffix in $|z| \leqslant r$ for which (10·21) is satisfied, $I_2(r)$ the number of those of odd suffix. Hence, if (10·20) is false, either

$$\overline{\lim} \frac{I_1(r)}{r} > 0 \quad \text{or} \quad \overline{\lim} \frac{I_2(r)}{r} > 0.$$

Take the first hypothesis, so that

$$\frac{I_1(R)}{R} > h \qquad (R = R_1, R_2, \ldots),$$

and, if $\gamma < \frac{1}{4}h$,

$$\nu(R) \geqslant I_1(\tfrac{1}{2}R) - \gamma R > \tfrac{1}{4}hR \qquad (R = 2R_1, 2R_2 \ \ldots).$$

Now take $\qquad t = \min(\beta, \frac{1}{4}h), \quad N = [Rt] = Rt'$,

R being one of the sequence $2R_1$, $2R_2$, ... Then $t' > \frac{1}{2}t$ and, by (10·23),

$$-\alpha\gamma + t' \log \frac{1}{t'} + t' \log \frac{2e}{\lambda} < -\tfrac{1}{2}\alpha\gamma.$$

Hence (10·22) gives

$$|g(z)| \leqslant M(R)(\tfrac{4}{5})^{\frac{1}{2}Rt+1} + 8\exp\{-\tfrac{1}{2}\alpha\gamma R\} \to 0 \quad \text{as } R = 2R_n \to \infty.$$

The contradiction implies that $I_1(r) = o(r)$, and similarly $I_2(r) = o(r)$.

Exceptional values. Pfluger and Pólya (**1**) have recently made an elegant application of Theorem 10 to prove the following result.

Theorem 14. *If an integral function of finite order p has a Borel exceptional value, its power series has a density equal to one of the fractions*

$$1/p, \quad 2/p, \quad \ldots, \quad p/p.$$

The density is defined as the limit of q_n/n as $n \to \infty$, q_n being the number of coefficients which do not vanish. The limit may not, of course, exist. Only functions of integral order p can possess a Borel exceptional value, and a will be such a value for $G(z)$ if and only if

$$G(z) = a + e^{bz^p} F(z),$$

where b is a constant other than zero, and $F(z)$ is an integral function of order less than p. All the fractions mentioned in Theorem 14 may arise, as is shown by the functions

$$e^{z^p}, \quad (1+z)\,e^{z^p}, \quad \ldots, \quad (1+z+z^2+\ldots+z^{p-1})\,e^{z^p},$$

all of which have the exceptional value 0.

Theorem 14 is deduced from the following more general result.

L_{402}. *Let p be a positive integer and let*

$$G(z) = e^{z^p} F(z) = c_0 + c_1 z + c_2 z^2 + \ldots,$$

where $F(z)$ is an integral function either of order less than p, or of minimum type of order p. Then the sequence

$$c_k, \quad c_{k+p}, \quad c_{k+2p}, \quad \ldots$$

has the density 1, *unless all its terms are zero.*

Theorem 14 is deduced by putting $k = 0, 1, \ldots, p-1$.

If L_{402} can be proved in the simplest case $p = 1$, $k = 0$, it can be proved generally. For, if $0 \leqslant k < p$ and $P = e^{2\pi i/p}$,

$$c_k + c_{k+p} z + c_{k+2p} z^2 + \ldots = e^z f(z),$$

where

$$f(z) = \frac{F(z^{1/p}) + P^{-k} F(Pz^{1/p}) + \ldots + P^{-(p-1)k} F(P^{p-1} z^{1/p})}{p z^{k/p}},$$

and this $f(z)$ satisfies (10·11). Thus, if L_{402} is true for $p = 1$, either $f(z)$ vanishes or its power series has density 1.

The case $p = 1$ is dealt with by writing

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}, \quad \phi(z) = \sum_{k=0}^{\infty} a_k \frac{z(z-1)\ldots(z-k+1)}{k!}.$$

The hypothesis is that $F(z)$ satisfies (10·11), and Theorem 10 shows that $\phi(z)$ also satisfies it. Now

$$e^z F(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{a_k z^k}{k!} \cdot \frac{z^l}{l!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} a_k = \sum_{n=0}^{\infty} \frac{z^n \phi(n)}{n!},$$

and the number of $\phi(0)$, $\phi(1)$, ..., $\phi(n)$ which vanish does not exceed the number of zeros of $\phi(z)$ in $|z| \leqslant n$. As $\phi(z)$ satisfies (10·11), this number is $o(n)$.

§ 11. The Newton-Gauss series and the cardinal series.

There are two formulae for interpolation at the points 0, $\pm n$, the Newton-Gauss series*

$$(11\cdot 1) \quad f(0) + \left\{ z \Delta f(0) + \frac{z(z-1)}{2!} \Delta^2 f(-1) \right\}$$
$$+ \left\{ \frac{z(z^2-1^2)}{3!} \Delta^3 f(-1) + \frac{z(z^2-1^2)(z-2)}{4!} \Delta^4 f(-2) \right\} + \dots$$

and the cardinal series

$$(11\cdot 2) \quad \frac{\sin \pi z}{\pi} \left[\frac{f(0)}{z} + \sum_{n=1}^{\infty} (-)^n \left\{ \frac{f(n)}{z-n} + \frac{f(-n)}{z+n} \right\} \right].$$

Steffensen (**1**) and E. T. Whittaker (**1**) showed that under certain conditions these series converge to the same sum and the precise relationship was discovered by Ferrar (**1**).

If (11·2) *is convergent,* (11·1) *converges to the same sum. If* (11·1) *is convergent,* (11·2) *is summable by the method of de la Vallée Poussin to the same sum.*

The class of functions represented by (11·1) has been discussed by Nörlund (**1**, **2**). It includes all integral functions satisfying (10·11). The cardinal series represents a much more restricted class of integral functions—all of order one.

* Whittaker and Robinson (**1**), Chapter III. From the theory of functions point of view the Gauss, Stirling and Bessel formulae are essentially the same. It will be noticed that the terms of the Gauss series have been bracketed in pairs.

Ferrar's result, in a more general form, can be deduced from the fundamental identity,

(11·3) $f(e_0)+(z-e_0)f(e_0,e_1)+\ldots+(z-e_0)\ldots$

$$(z-e_{n-1})f(e_0,e_1,\ldots,e_n)=\phi(z)\sum_{i=0}^{n}\frac{f(e_i)}{\phi'(e_i)(z-e_i)},$$

where
$$\phi(z)=\prod_{i=0}^{n}\left(1-\frac{z}{e_i}\right),$$

(11·4)
$$f(e_0,e_1)=\frac{f(e_1)-f(e_0)}{e_1-e_0},$$
$$f(e_0,e_1,e_2)=\frac{f(e_1,e_2)-f(e_0,e_1)}{e_2-e_0},\quad \ldots.$$

The left-hand side of (11·3) is Newton's divided difference interpolation formula* and the right-hand side is Lagrange's formula. Their identity follows from the fact that both are polynomials of degree n which take any given values $f(e_0)$, $f(e_1)$, ..., $f(e_n)$ at the points $e_0, e_1, \ldots, e_n$.

Let $c_1, c_2, \ldots$ be a strictly increasing sequence of positive numbers such that Σc_n^{-2} converges and let

(11·5)
$$H(z)=z\prod_{i=1}^{\infty}\left(1-\frac{z^2}{c_i^2}\right),\quad H_n(z)=z\prod_{i=1}^{n}\left(1-\frac{z^2}{c_i^2}\right).$$

(11·3) gives

(11·6) $f(0)+zf(0,c_1)+\ldots+z(z^2-c_1^2)\times$

$$\ldots(z^2-c^2_{n-1})(z-c_n)f(0,c_1,-c_1,\ldots,c_n,-c_n)$$

$$=H_n(z)\left[\frac{f(0)}{z}+\sum_{m=0}^{n}\left\{\frac{f(c_m)}{H'_n(c_m)(z-c_m)}+\frac{f(-c_m)}{H'_n(c_m)(z+c_m)}\right\}\right]$$

$$=H_n(z)\left[\frac{f(0)}{z}+\sum_{m=0}^{n}\phi(n,m)\right.$$

$$\left.\times\left\{\frac{f(c_m)}{H'(c_m)(z-c_m)}+\frac{f(-c_m)}{H'(c_m)(z+c_m)}\right\}\right],$$

where

(11·7)
$$\phi(n,m)=\frac{H'(c_m)}{H_n'(c_m)}=\prod_{i=n+1}^{\infty}\left(1-\frac{c_m^2}{c_i^2}\right).$$

* Whittaker and Robinson (**1**), Chapter II.

Let $\lambda(n)$ be a decreasing positive function such that $\Sigma\lambda(n)$ converges and let

$$\psi(n, m) = \prod_{i=n+1}^{\infty} \left\{1 - \frac{\lambda(i)}{\lambda(m)}\right\} \qquad (m = 0, 1, \ldots, n).$$

If
$$U_n = \sum_{m=0}^{n} \psi(n, m) u_m \to U \quad \text{as } n \to \infty,$$

the series Σu_n may be said to be *summable* $\{$V.P. $\lambda(n)\}$ to sum U. This method of summation sums every convergent series to the correct value.* If $\lambda(n) = 1/n^2$,

$$\psi(n, m) = \prod_{i=n+1}^{\infty} \left(1 - \frac{m^2}{i^2}\right) = \frac{n!\, n!}{(n-m)!\,(n+m)!},$$

so that summability (V.P. $1/n^2$) is identical with the method used by de la Vallée Poussin (**1**) to sum Fourier series. The application to interpolation follows from (11·6).

Theorem 15. *If the series*

$$(11\cdot8) \quad H(z)\left[\frac{f(0)}{z} + \sum_{n=0}^{\infty} \left\{\frac{f(c_n)}{H'(c_n)(z-c_n)} + \frac{f(-c_n)}{H'(c_n)(z+c_n)}\right\}\right]$$

is convergent, the series

$$(11\cdot9) \quad f(0) + \{zf(0, c_1) + z(z-c_1) f(0, c_1, -c_1)\} + \{z(z^2 - c_1^2)$$
$$\times f(0, c_1, -c_1, c_2) + z(z^2 - c_1^2)(z - c_2) f(0, c_1, -c_1, c_2, -c_2)\} + \ldots$$

converges to the same sum. If (11·9) *is convergent,* (11·8) *is summable* (V.P. $1/c_n^2$) *to the same sum.*

Ferrar's result is the case $c_n = n$. It follows from it that, if the sequence $\{f(n)\}$ is bounded for all positive and negative values of n, the series (11·1), (11·2) are equivalent, being either both divergent or both convergent to the same sum. For, if (11·1) is convergent, (11·2) is summable by the method of de la Vallée Poussin and hence by the Abel limit;† but the general term of (11·2) is $O\left(\frac{1}{n}\right)$ and so, by Littlewood's converse of Abel's theorem, (11·2) is convergent (**1**).

* Necessary and sufficient conditions that this should be the case will be found in Dienes (**1**), 396. It is easy to verify that they are satisfied here.

† Cf. Gronwall (**1**).

*The Fourier theory.** The cardinal series is closely connected with certain aspects of the theory of Fourier series and integrals. The first result in this direction, due to E. T. Whittaker (**1**), asserts that, under sufficiently strong conditions, the cardinal function can be resolved by Fourier's integral and that all components of period less than 2 will be absent. The conditions can be relaxed considerably by introducing Stieltjes integrals, as originally defined by Stieltjes; that is to say,

$$(11\cdot10) \qquad \int_a^b f(x)\,dg(x) = \lim_{\lambda\to 0} \sum_{r=1}^{n} f(\xi_r)\{g(x_r) - g(x_{r-1})\},$$

where $\lambda = \max(x_r - x_{r-1})$, and ξ_r is any point of (x_{r-1}, x_r). We need two properties of the integral.

L$_{403}$. *If*

(i) $V_a{}^b f_p(x) + |f_p(b)| < K \qquad$ (all p);

(ii) $f_p(x) \to f(x) \qquad (a \leqslant x \leqslant b)$;

(iii) $g(x)$ *is continuous in* (a, b);

then

$$(11\cdot11) \qquad \int_a^b f_p(x)\,dg(x) \to \int_a^b f(x)\,dg(x).$$

L$_{404}$. *If* $\int_a^b g(x)\,d\phi(x)$ *exists and* $f(x)$ *is bounded in* (a, b),

$$(11\cdot12) \qquad \int_a^b f(x)\,d\psi(x) = \int_a^b f(x)\,g(x)\,d\phi(x),$$

where
$$\psi(x) = \int_a^x g(t)\,d\phi(t),$$

whenever either integral in (11·12) *exists.*

The first of these is due to Hahn (**1**, 84), the second to Hyslop (**1**), extending a result of Carleman.

Let us say that the series

$$(11\cdot13) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

* In this subsection all the numbers concerned are real.

is a "Fourier-Stieltjes" series if there is a continuous function $F(x)$ such that

$$(11\cdot 14)\qquad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} dF(x),\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\,dF(x),$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\,dF(x).$$

L_{405}. *The necessary and sufficient condition that* (11·13) *should be a Fourier-Stieltjes series is that*

$$\sum_{n=1}^{\infty}\frac{1}{n}(a_n\sin nx-b_n\cos nx)$$

should be the Fourier series of a continuous function $G(x)$, *the functions* $F(x)$, $G(x)$ *being connected by the equation*

$$F(x)=G(x)+\tfrac{1}{2}a_0x.$$

To prove the necessity of the condition, integrate by parts in (11·14). Then

$$a_n=\left[\frac{\cos nx\,F(x)}{\pi}\right]_{-\pi}^{\pi}+\frac{n}{\pi}\int_{-\pi}^{\pi}F(x)\sin nx\,dx$$
$$=(-)^n a_0+\frac{n}{\pi}\int_{-\pi}^{\pi}F(x)\sin nx\,dx$$
$$(11\cdot 15)\qquad =\frac{n}{\pi}\int_{-\pi}^{\pi}G(x)\sin nx\,dx,$$

since $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}dF(x)=\frac{F(\pi)-F(-\pi)}{\pi}.$$

Similarly

$$(11\cdot 16)\qquad b_n=-\frac{n}{\pi}\int_{-\pi}^{\pi}G(x)\cos nx\,dx.$$

Thus $-b_n/n$, a_n/n are the Fourier coefficients of $G(x)$, and the latter is evidently continuous. Again, the condition is sufficient. For, if it is satisfied, (11·15), (11·16) are true and, on integrating by parts, we obtain (11·14).

Returning to the cardinal function, assume that $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ is a sequence such that

$$(11{\cdot}17) \qquad \sum_{n=1}^{\infty} \frac{1}{n}(|a_n|+|a_{-n}|)<\infty;$$

$$\sum_{n=1}^{\infty} \frac{1}{n}(a_n+a_{-n})\sin \pi nt, \quad \sum_{n=1}^{\infty} \frac{1}{n}(a_n-a_{-n})\cos \pi nt$$

then converge to continuous functions, so, by L_{405}, there are continuous functions $\Phi(t)$, $\Psi(t)$, such that

$$a_0=\int_0^1 d\Phi(t), \quad \tfrac{1}{2}(a_n+a_{-n})=\int_0^1 \cos \pi nt\, d\Phi(t),$$

$$\tfrac{1}{2}(a_n-a_{-n})=\int_0^1 \sin \pi nt\, d\Psi(t),$$

whence, for all the values of n,

$$(11{\cdot}18) \qquad a_n=\int_0^1 \{\cos \pi nt\, d\Phi(t)+\sin \pi nt\, d\Psi(t)\}.$$

Consider now the Fourier series of $\cos \pi xt$, $\sin \pi xt$, considered as functions of t, namely

$$(11{\cdot}19) \quad \begin{cases} \dfrac{\sin \pi x}{\pi}\left\{\dfrac{1}{x}+\displaystyle\sum_{n=1}^{\infty}(-)^n \cos \pi nt\left(\dfrac{1}{x-n}+\dfrac{1}{x+n}\right)\right\}, \\ \dfrac{\sin \pi x}{\pi}\displaystyle\sum_{n=1}^{\infty}(-)^n \sin \pi nt\left(\dfrac{1}{x-n}-\dfrac{1}{x+n}\right). \end{cases}$$

As $\cos \pi xt$, $\sin \pi xt$ are bounded and of bounded variation in $a \leqslant t \leqslant b$, condition (i) of L_{403} is satisfied, where $f_p(x)$ denotes the pth partial Cesàro sum of either of the Fourier series.* Again, since the functions are continuous, $f_p(t)$ converges to $\cos \pi xt$, $\sin \pi xt$, and condition (ii) is satisfied. Hence, given any function of the form

$$(11{\cdot}20) \qquad f(x)=\int_0^1 \{\cos \pi xt\, d\Phi(t)+\sin \pi xt\, d\Psi(t)\},$$

where $\Phi(t)$, $\Psi(t)$ are continuous functions, it is permissible to substitute the series (11·19) in the integral and then to integrate

* Hobson (1), 560, 580.

term by term, the resulting series being summable $(C, 1)$, i.e. the series

$$(11\cdot21) \qquad \frac{\sin \pi x}{\pi}\left[\frac{f(0)}{x}+\sum_{n=1}^{\infty}(-)^n\left\{\frac{f(n)}{x-n}+\frac{f(-n)}{x+n}\right\}\right]$$

is summable $(C, 1)$ and its sum is $f(x)$.

If (11·17) is satisfied, the cardinal series is evidently absolutely convergent and a comparison of (11·18), (11·20) shows that $f(n)=a_n$. The following result has therefore been established (**2**).

Theorem 16. *Given any function $f(x)$ of the form*

$$(11\cdot22) \qquad \int_0^1 \{\cos \pi xt\, d\Phi(t) + \sin \pi xt\, d\Psi(t)\} \quad [\Phi(t), \Psi(t) \textit{ continuous functions}],$$

the series (11·21) *is summable* $(C, 1)$ *and its sum is* $f(x)$.

If (11·17) *is satisfied, the cardinal series*

$$(11\cdot23) \qquad \frac{\sin \pi x}{\pi}\left\{\frac{a_0}{x}+\sum_{n=1}^{\infty}(-)^n\left(\frac{a_n}{x-n}+\frac{a_{-n}}{x+n}\right)\right\}$$

is absolutely convergent and its sum is of the form (11·22).

This result enables us to deal with a remarkable property of the cardinal series discovered by Ferrar (**1**, **2**), and called by him its "consistency".* Its genesis is as follows. Suppose that we are given n points and that we draw through them, by means of Newton's formula, a curve of degree $n-1$. On this curve take any other n points. Since only one curve of degree $n-1$ can be drawn through n points, the curve obtained by applying Newton's formula to the new points will be the same as the first curve. The interpolation curve can therefore reproduce itself from any n points on it, unless of course these lie on a curve of lower degree. Ferrar showed that the cardinal series has a property of the same nature. If $C(x)$ is the cardinal series for the points $(a+nw, a_n)$, i.e.

$$C(x)=\frac{w}{\pi}\sum a_n \frac{\sin \frac{\pi}{w}(x-a-nw)}{x-a-nw},$$

* The corresponding property of the Gregory-Newton series had been investigated by Nörlund (**1**).

and if $w' < w$, then the cardinal series $C_1(x)$ for the points $\{b+nw', C(b+nw')\}$ is identical with $C(x)$, provided that (**11·17**) is satisfied. The case $w'=w$ is more difficult. Ferrar pointed out that the series inversion formulae due to Titchmarsh (**1**, **2**) were equivalent to asserting that $C_1(n)=C(n)$, provided that $\Sigma |a_n|^p$ is convergent for some $p>1$; and he proved that, on this hypothesis, $C_1(x)=C(x)$ for all x. The result which follows asserts consistency on a weaker hypothesis (**2**).

Theorem 17. *If*

$$\sum_{n=2}^{\infty} (|a_n|+|a_{-n}|)\frac{\log n}{n} < \infty, \tag{11·24}$$

so that the cardinal series (11·23) *is absolutely convergent, and* λ *is any real number,*

$$C(x)=\frac{\sin \pi(x-\lambda)}{\pi}\sum_{n=-\infty}^{\infty}(-)^n\frac{C(n+\lambda)}{x-\lambda-n}, \tag{11·25}$$

the series on the right being absolutely convergent. Moreover

$$\begin{aligned}|C(\lambda)|+\sum_{n=1}^{\infty}\frac{1}{n}\{|C(n+\lambda)|+|C(-n+\lambda)|\} \\ < A(\lambda)\left\{|a_0|+|a_1|+|a_{-1}|+\sum_{n=2}^{\infty}(|a_n|+|a_{-n}|)\frac{\log n}{n}\right\},\end{aligned} \tag{11·26}$$

where $A(\lambda)$ *depends only on* λ.

Theorem 16 shows that $C(x)$ is of the form

$$\int_0^1 \{\cos \pi xt\, d\Phi(t)+\sin \pi xt\, d\Psi(t)\},$$

so that, making use of L_{404},

$$\begin{aligned} C(x+\lambda) &= \int_0^1 [\cos \pi xt\{\cos \pi\lambda t\, d\Phi(t)+\sin \pi\lambda t\, d\Psi(t)\} \\ &\qquad + \sin \pi xt\{-\sin \pi\lambda t\, d\Phi(t)+\cos \pi\lambda t\, d\Psi(t)\}] \\ &= \int_0^1 [\cos \pi xt\, d\Phi_1(t)+\sin \pi xt\, d\Psi_1(t)],\end{aligned}$$

where
$$\Phi_1(x)=\int_0^x \{\cos \pi\lambda t\, d\Phi(t)+\sin \pi\lambda t\, d\Psi(t)\},$$

$$\Psi_1(x)=\int_0^x \{-\sin \pi\lambda t\, d\Phi(t)+\cos \pi\lambda t\, d\Psi(t)\}.$$

$C(x+\lambda)$ is thus an integral of the form (11·22), and the first part of Theorem 16 shows that

$$\frac{\sin \pi x}{\pi}\left[\frac{C(\lambda)}{x}+\sum_{n=1}^{\infty}(-)^n\left\{\frac{C(n+\lambda)}{x-n}+\frac{C(-n+\lambda)}{x+n}\right\}\right]$$

is summable $(C, 1)$ to $C(x+\lambda)$; or, changing x to $x-\lambda$, that

$$\frac{\sin \pi(x-\lambda)}{\pi}\left[\frac{C(\lambda)}{x-\lambda}+\sum_{n=1}^{\infty}(-)^n\left\{\frac{C(n+\lambda)}{x-\lambda-n}+\frac{C(-n+\lambda)}{x-\lambda+n}\right\}\right]$$

is summable $(C, 1)$ to $C(x)$.

To prove (11·26), take for simplicity the case $0<\lambda<1$. Then

$$|C(n+\lambda)|<\frac{1}{\pi}|\sin \pi\lambda|\sum_{r=-\infty}^{\infty}\frac{|a_r|}{|n+\lambda-r|},$$

so that

$$\sum_{n=1}^{N}\frac{1}{n}|C(n+\lambda)|<\frac{1}{\pi}|\sin \pi\lambda|\sum_{n=1}^{N}\frac{1}{n}\sum_{r=-\infty}^{\infty}\frac{|a_r|}{|n+\lambda-r|}$$

$$=\frac{1}{\pi}|\sin \pi\lambda|\left\{\sum_{r=0}^{\infty}|a_{-r}|\sum_{n=1}^{N}\frac{1}{n(n+\lambda+r)}+\sum_{r=1}^{\infty}|a_r|\sum_{n=1}^{N}\frac{1}{n|n+\lambda-r|}\right\}.$$

Now, if $r\geqslant 1$,

$$\sum_{n=1}^{N}\frac{1}{n(n+\lambda+r)}<\sum_{n=1}^{N}\frac{1}{n(n+r)}=\frac{1}{r}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+r}\right)$$

$$<\frac{1}{r}\sum_{n=1}^{r}\frac{1}{n}<\frac{1+\log r}{r},$$

and

$$\sum_{n=1}^{N}\frac{1}{n|n+\lambda-r|}=\sum_{n=1}^{r-1}\frac{1}{n(r-n-\lambda)}+\sum_{n=r}^{N}\frac{1}{n(n+\lambda-r)}$$

$$=\frac{1}{r-\lambda}\sum_{n=1}^{r-1}\left(\frac{1}{n}+\frac{1}{r-n-\lambda}\right)+\sum_{n=0}^{N-r}\frac{1}{(n+r)(n+\lambda)}$$

$$<\frac{1}{r-\lambda}\left\{\sum_{n=1}^{r-1}\frac{1}{n}+\frac{1}{1-\lambda}+\sum_{k=1}^{r-2}\frac{1}{k}\right\}+\sum_{n=0}^{N-r}\frac{1}{(n+r)(n+\lambda)}$$

$$<\frac{1}{r-\lambda}\left\{2+\log r+\frac{1}{1-\lambda}\right\}+A_1(\lambda)\frac{\log r}{r}.$$

$|C(-n+\lambda)|$ can be discussed similarly. It is not difficult to see that $A(\lambda)$ is a continuous function of λ.

The consistency theorem is the analogue for series of the well-known inversion theorem that, under certain conditions,

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt,$$

the integral being defined as a principal value when $t = x$, implies that

$$f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(t)}{t-x}\,dt$$

almost everywhere. Titchmarsh (**1**) has in fact deduced this result, on the hypothesis

$$\int_{-\infty}^{\infty} |f(x)|^p\,dx < \infty \qquad (p > 1),$$

from his inversion formulae for series referred to above.

Theorems 16, 17 do not apply to the general partial fraction series (11·8). The cardinal series is particularly favoured because the integral function $H(z)$ associated with it happens to be $\sin \pi z$ and this has several very special properties. Thus it is orthogonal relative to its own zeros,

$$(11{\cdot}27) \qquad \int_0^1 H(c_n t)\, H(c_m t)\, \phi(t)\, dt = 0 \qquad (n \neq m),$$

with $c_n = n$, $\phi(t) = 1$, and this gives rise to Theorem 16. A similar theorem holds if $c_n = j_n$, the nth positive zero of $J_0(z)$. For then $H(z) = zJ_0(z)$, and (11·27) is true with $\phi(t) = 1/t$. Again, the proof of Theorem 17 depends on the fact that $\sin \pi z$ has an addition theorem, and this property is not possessed by any other $H(z)$, the so-called addition theorem for Bessel functions not being an adequate substitute.

The cardinal series has been extensively studied. See Ferrar (**1**, **2**, **3**), Copson (**1**, **2**, **3**), Ogura (**1**), Pólya (**2**), Miss Cartwright (**1**, **2**), Tschakaloff (**1**) for investigations concerning its connection with Hardy's "m-functions", its zeros, its integral representations, etc. Other references will be found in these papers.

CHAPTER V

INTERPOLATION AT THE LATTICE POINTS

§ 12. The two-dimensional cardinal series.

Interpolation at the lattice points can be effected by means of a series analogous to (11·2). To replace $\sin \pi z$ we need an integral function with simple zeros at the lattice points. Such a function is the Weierstrass sigma function formed with periods 1, i,

$$(12\cdot1) \qquad \sigma(z) = e^{\pi z^2/2} \vartheta_1(\pi z \mid i)/\vartheta_1'(0)\,\pi.$$

By the properties of the ϑ-function,*

$$(12\cdot2) \quad \sigma(z+m+ni) = (-)^{m+n+mn}\,\sigma(z)\exp\{\tfrac{1}{2}\pi(m^2+n^2)+\pi(m-ni)z\},$$

so that

$$(12\cdot3) \qquad \sigma'(m+ni) = (-)^{m+n+mn}\exp\{\tfrac{1}{2}\pi(m^2+n^2)\}.$$

$\mathbf{L}_{501}$. *Let $f(z)$ be an integral function which satisfies the condition*

$$(12\cdot4) \qquad \overline{\lim_{r\to\infty}}\, \frac{\log M(r)}{r^2} < \frac{\pi}{2}.$$

Then

$$(12\cdot5) \quad f(z) = \sigma(z) \sum_{m,\,n=-\infty}^{\infty} \frac{f(m+ni)}{z-m-ni}(-)^{m+n+mn}\,e^{-\frac{1}{2}\pi(m^2+n^2)}.$$

Let K_p denote the square whose corners are $(\pm 1 \pm i)(p+\tfrac{1}{2})$. Then, if z is a point inside K_p other than one of the points $m+ni$,

$$\frac{f(z)}{\sigma(z)} + \sum_{K_p} \frac{f(m+ni)}{\sigma'(m+ni)(m+ni-z)} = \frac{1}{2\pi i}\int_{K_p} \frac{f(\zeta)}{\sigma(\zeta)(\zeta-z)}\,d\zeta.$$

It follows from (12·2), (12·4) that the integral on the right tends to zero as $p\to\infty$, and from (12·3), (12·4) that the series

$$\sum_{m,\,n=-\infty}^{\infty} \left|\frac{f(m+ni)}{\sigma'(m+ni)}\right| = \sum_{m,\,n=-\infty}^{\infty} |f(m+ni)|\,e^{-\frac{1}{2}\pi(m^2+n^2)}$$

converges. These results establish $\mathbf{L}_{501}$ (**3**, **6**).

* Whittaker and Watson (**1**), 464.

It will be noticed that (12·5) represents all integral functions of sufficiently slow growth, so in this respect it is analogous to (11·1) rather than to (11·2). It came to light during an attempt to settle a conjecture of Littlewood, that an integral function of order less than 2 cannot be bounded at the lattice points unless it is a constant. This was established by the methods of the next section, but Pólya (**6**) has recently found an elegant method of completing the original argument.

L_{502}. *Let $f(z)$ be an integral function which satisfies the conditions*

$$\frac{\log M(r)}{r^2} \to 0 \quad \text{as } r \to \infty; \tag{12·6}$$

$$|f(m+ni)| < K \qquad (m, n = 0, \pm 1, \pm 2, \ldots), \tag{12·7}$$

then $f(z)$ is a constant.

For $f(kz)$, where k is any integer, satisfies (12·4), and so

$$f(z) = f\left(k\frac{z}{k}\right) = k\sigma\left(\frac{z}{k}\right) \sum_{m,n=-\infty}^{\infty} \frac{f(km+kni)}{z-k(m+ni)} (-)^{m+n+mn} e^{-\frac{1}{2}\pi(m^2+n^2)}.$$

On making $k \to \infty$, using (12·7), the only term which survives is that with $m = 0$, $n = 0$, and we obtain

$$f(z) = z\sigma'(0)\frac{f(0)}{z} = f(0).$$

§ 13. The "flat" regions of integral and meromorphic functions.

Results such as L_{502} can be proved by an entirely different method. This consists in showing that an integral function of finite order satisfies the inequality*

$$\log|f(z)| > h \log M(r) \qquad (|z| = r) \tag{13·1}$$

in regions as large, though not as numerous, as those in which it

* This result is not connected with the important theorems of Wiman and Valiron (**1**, Chapter IV) on the regions, in the neighbourhood of points at which $f(z)$ assumes its maximum modulus, in which the inequality $|f(z)| > hM(r)$ is satisfied. It is of the same nature as the theorem of Littlewood, Wiman and Valiron that a function of order $\rho < 1$ satisfies the inequality

$$\log|f(z)| > (\cos \pi\rho - \epsilon) \log M(r)$$

on arbitrarily large circles $|z| = r$. See Valiron (**1**, 128), Besicovitch (**1**).

satisfies the inequality $|f(z)| > 0$. The proof is based on the following lemmas:

L$_{503}$. *Let $r_1, r_2, \ldots$ be an increasing sequence of positive numbers. Divide the real positive axis into segments of given length λ and mark the points $r_1, r_2, \ldots$ on it. Now shade every segment containing an r, and its two neighbours. Of the remaining segments shade every one whose two neighbours on the right contain two or more r's. Then every segment whose three neighbours on the right contain three or more r's, and so on. Perform the same process for neighbouring segments on the left. Let $n(r)$ denote the number of points $r_1, r_2, \ldots$ in $(0, r)$. Then, if*

$$\frac{n(r)}{r} \to 0 \quad \text{as } r \to \infty, \tag{13·2}$$

almost every segment is unshaded.

By the last statement we mean that

$$\frac{N_s}{N} \to 0 \quad \text{as } N \to \infty, \tag{13·3}$$

where N_s is equal to the number of shaded segments among the first N.

Suppose that this is false. Then there is a number h, $(0 < h < 1)$, such that

$$\frac{N_s}{N} > h, \tag{13·4}$$

for arbitrarily large values of N. Find r_0 so that

$$\frac{n(r)}{r} < \frac{h}{6\lambda} \qquad (r \geqslant r_0), \tag{13·5}$$

and let N_1 be a number greater than $2\lambda r_0$ for which (13·4) is satisfied.

Let $r_1, r_2, \ldots, r_m$ be the r's in the first N_1 segments. Suppose now that the shading process is carried out in two stages. First mark in $r_1, r_2, \ldots, r_m$ only and perform the process for these points, and then mark in the other r's and complete the process. It is easy to see, by considering simple cases, that at most $3m$ segments will be shaded in the first stage. Let k additional segments among the first N_1 be shaded in the second stage. In the most unfavourable case (i.e. the case implying the least number of r's) these will be the last k. Suppose that this is so. Then, for some p, the $p+1$ segments immediately succeeding the N_1th must contain at least $k+p$ of the points r_s. By (13·4),

$$3m+k>hN_1.$$

But $$n\{(N_1+p+1)\lambda\}\geqslant m+k+p>\frac{h}{3}N_1+p>\frac{h}{3}(N_1+p),$$

so that $$n\{(N_1+p)\,2\lambda\}>\frac{h}{3}(N_1+p),$$

or $$n(r)>\frac{h}{6\lambda}r,$$

for $r=(N_1+p)\,2\lambda$; which contradicts (13·5), since $r>r_0$.

L_{504}. *Divide the z-plane into squares of side λ by drawing lines parallel to the axes and mark points $c_1, c_2, \ldots$ on it. Shade every square containing a c and the eight neighbouring squares. Of the remaining squares shade every one whose twenty-four neighbours contain eight or more c's, and, generally, every square whose $4q(q+1)$ neighbours contain $4q(q-1)$ or more c's. Let $c(r)$ denote the number of c's for which $|c_s|\leqslant r$. Then, if*

$$\frac{c(r)}{r^2}\to 0 \quad \textit{as } r\to\infty,$$

almost every square is unshaded.

The conclusion of the lemma means that

$$\frac{N_s}{N}\to 0 \quad \text{as } N\to\infty,$$

where N_s is equal to the number of shaded squares among the N squares nearest the origin. The proof is similar to that of L_{503}.

L_{505}. *Let $f(z)$ be an integral function of order less than or equal to 2 satisfying the condition*

$$n(r)=o(r^2), \tag{13·6}$$

and let λ, δ be given positive numbers. Then, for almost every square of side λ, drawn as above,

$$|\log f(z_1)-\log f(z_2)|<r^{1+\delta}, \tag{13·7}$$

where z_1, z_2 are any points of the square, and r is the distance of the centre of the square from the origin.

$n(r)$ denotes the number of zeros a_s of $f(z)$ for which $r_s = |a_s| \leqslant r$. Multiple zeros are counted multiply.

There is no loss of generality in taking $f(0) = 1$ so that $f(z)$ is of the form

$$(13\cdot8)\quad f(z) = \exp(kz + lz^2) \prod_{s=1}^{\infty} \left\{ \left(1 - \frac{z}{a_s}\right) \exp\left(\frac{z}{a_s} + \frac{1}{2}\frac{z^2}{a_s^2}\right) \right\}.$$

Mark the points $a_1, a_2, \ldots$ on the z-plane and shade as in the last lemma. Then almost every square is unshaded. Take points z_1, z_2 in an unshaded square and denote by α_s the smaller of $|a_s - z_1|, |a_s - z_2|$. Then

$$\begin{aligned}
&|\log f(z_1) - \log f(z_2)| \\
&\leqslant |k|\,|z_1 - z_2| + |l|\,|z_1^2 - z_1^2| \\
&\qquad + \sum_{s=1}^{\infty} \left| \log\left(1 - \frac{z_1 - z_2}{a_s - z_2}\right) + \frac{z_1 - z_2}{a_s} + \frac{z_1^2 - z_2^2}{2a_s^2} \right| \\
&< Kr\,|z_1 - z_2| + \sum_{s=1}^{\infty} \left| -\frac{z_1 - z_2}{a_s - z_2} - \frac{(z_1 - z_2)^2}{2(a_s - z_2)^2} + \frac{z_1 - z_2}{a_s} + \frac{z_1^2 - z_2^2}{2a_s^2} \right| \\
&< Kr\,|z_1 - z_2| + |z_1 - z_2| \sum_{s=1}^{\infty} \left| -\frac{z_1 z_2}{a_s (a_s - z_2)^2} + \frac{z_2^2 (z_1 + z_2)}{2a_s^2 (a_s - z_2)^2} \right| \\
&< K\lambda r + 2\lambda r^2 \sum_{s=1}^{\infty} \frac{1}{r_s \alpha_s^2} + 2\lambda r^3 \sum_{s=1}^{\infty} \frac{1}{r_s^2 \alpha_s^2}.
\end{aligned}$$

Take ϵ so that $1 - \frac{1}{2}\delta < \epsilon < 1$, and consider the square P of side $(2p+1)\lambda$, where $p = [r^\epsilon]$, symmetrically disposed about the square containing z_1, z_2. For points a_s inside P, $r_s > \frac{1}{2}r$, so that

$$\sum_P \frac{1}{r_s \alpha_s^2} < \frac{2}{r} \sum_P \frac{1}{\alpha_s^2}, \quad \sum_P \frac{1}{r_s^2 \alpha_s^2} < \frac{4}{r^2} \sum_P \frac{1}{\alpha_s^2}.$$

Now, since the square containing z_1, z_2 is unshaded, there are no points a_s in the first ring of squares round it, nor more than seven in the second ring, nor more than twenty-three in the second and third ring, etc. Thus

$$\begin{aligned}
\sum_P \frac{1}{\alpha_s^2} &\leqslant \frac{7}{\lambda^2} + \frac{23 - 7}{(2\lambda)^2} + \ldots + \frac{4p(p-1) - 4(p-1)(p-2)}{\{(p-1)\lambda\}^2} \\
&< \frac{8}{\lambda^2} \sum_{q=1}^{p-1} \frac{1}{q} < \frac{8}{\lambda^2} \log r
\end{aligned}$$

Again, applying Hölder's inequality to the sum of the remaining terms of the first series, we have

$$\sum_{CP} \frac{1}{r_s \alpha_s^2} \leqslant \left(\sum_{CP} r_s^{\epsilon-3}\right)^{1/(3-\epsilon)} \left(\sum_{CP} \alpha_s^{2(\epsilon-3)/(2-\epsilon)}\right)^{(2-\epsilon)/(3-\epsilon)}$$

$$< K\left(\int_{r^\epsilon}^{\infty}\int_0^{2\pi} u^{2(\epsilon-3)/(2-\epsilon)}\, u\, du\, d\theta\right)^{(2-\epsilon)/(3-\epsilon)}$$

$$< K r^{-2\epsilon/(3-\epsilon)}.$$

Moreover, if $\qquad \beta > 2(\delta + 2\epsilon - 2)^{-1},$

$$\sum_{CP} \frac{1}{r_s^2 \alpha_s^2} \leqslant \left(\sum_{CP} r_s^{2\beta/(1-\beta)}\right)^{(\beta-1)/\beta} \left(\sum_{CP} \alpha_s^{-2\beta}\right)^{1/\beta}$$

$$< K\left(\int_{r^\epsilon}^{\infty}\int_0^{2\pi} u^{-2\beta}\, u\, du\, d\theta\right)^{1/\beta} < K r^{(2-2\beta)\epsilon/\beta}.$$

Since $\qquad 2 - 2\epsilon/(3-\epsilon) < 1 + \delta, \quad 3 + (2-2\beta)\epsilon/\beta < 1 + \delta,$

these results establish the lemma.

Upper and lower density. Let E be a plane measurable set of points, and let $m(E, r)$ be the (plane) measure of the part of E contained in the circle of centre the origin, radius r. We call

$$(13\cdot 9) \qquad \underline{D}E = \varliminf_{r\to\infty} \frac{m(E,r)}{\pi r^2}, \quad \overline{D}E = \varlimsup_{r\to\infty} \frac{m(E,r)}{\pi r^2}$$

the *lower* and *upper densities* of E. If they are equal, the common value will be called the *density* of E.

L_{506}. *If E_1 is any set and E_2 a set of density unity, then*

$$\underline{D}(E_1 E_2) = \underline{D}E_1, \quad \overline{D}(E_1 E_2) = \overline{D}E_1.$$

The proof is immediate.*

Meromorphic functions of order 2. Let $L(\zeta, d)$, $U(\zeta, d)$ denote the bounds of an integral or meromorphic function in the circle $|z - \zeta| \leqslant d$, and let $n(r, 0)$, $n(r, \infty)$ denote, as usual, the number of zeros and the number of poles of the function in the circle $|z| \leqslant r$.

* The density of a linear set has been defined by Besicovitch (**1**).

L_{507}. *Let $f(z)$ be an integral or meromorphic function of order less than or equal to 2 satisfying the condition*

$$n(r,0)+n(r,\infty)=o(r^2), \tag{13·10}$$

and let d, δ be given positive numbers. Then the values of ζ for which

$$U(\zeta,d)\leqslant L(\zeta,d)\,e^{r^{1+\delta}} \qquad (|\zeta|=r) \tag{13·11}$$

form a set of unit density.

$f(z)$ can be expressed in the form* $f_1(z)/f_2(z)$, where $f_1(z), f_2(z)$ are integral functions satisfying the conditions of L_{505}. Given ϵ, take λ so that

$$\frac{\lambda^2-(\lambda-2d)^2}{\lambda^2}<\epsilon \tag{13·12}$$

and divide the z-plane into squares of side λ. If ζ is in the square of side $\lambda-2\alpha$ concentric with a square of side λ, the inequalities

$$|\log f_1(z)-\log f_1(\zeta)|<\tfrac{1}{4}r^{1+\delta}, \quad |\log f_2(z)-\log f_2(\zeta)|<\tfrac{1}{4}r^{1+\delta},$$

where $|z-\zeta|\leqslant d$, are satisfied for almost every such square. Thus, for almost every square,

$$\begin{aligned}
&\log U(\zeta,d)-\log L(\zeta,d)\\
&\quad=\log\left|\frac{f_1(z_1)}{f_2(z_1)}\right|-\log\left|\frac{f_1(z_2)}{f_2(z_2)}\right| \qquad (z_1,\ z_2 \text{ points in } |z-\zeta|\leqslant d)\\
&\quad\leqslant |\log|f_1(z_1)|-\log|f_1(z_2)||+|\log|f_2(z_1)|-\log|f_2(z_2)||\\
&\quad\leqslant |\log f_1(z_1)-\log f_1(z_2)|+|\log f_2(z_1)-\log f_2(z_2)|\\
&\quad< r^{1+\delta}.
\end{aligned}$$

If r is sufficiently large, the squares (of side λ) for which this holds cover a fraction greater than $1-\epsilon$ of the circle of radius r, and, by (13·12), the measure of the set of ζ for which the inequality holds is greater than $(1-\epsilon)^2\pi r^2$. This is the result stated.

If $f(z)$ is of order $\rho<2$, more precise information, which will be useful later, can be obtained.

* Nevanlinna (**1**), 40.

L$_{508}$. *If $f(z)$ is an integral or meromorphic function of order $\rho\,(1 \leqslant \rho < 2)$, and $\sigma > \frac{1}{2}\rho$, the values of ζ for which*

$$U(\zeta, d) \leqslant L(\zeta, d)\, e^{r^\sigma} \tag{13·13}$$

form a set of unit density. If $f(z)$ is of genus zero and δ is any given positive number, (13·13) *can be replaced by*

$$U(\zeta, d) \leqslant L(\zeta, d)\, r^\delta. \tag{13·14}$$

This is proved by a similar method, L$_{503}$ taking the place of L$_{504}$ in the case of a function of genus zero.

L$_{509}$. *Let $\phi(x)$, $\omega(x)$ be real functions satisfying the conditions*

$$0 < \omega(x) \leqslant \omega(x') \qquad (x' > x \geqslant a), \tag{13·15}$$

$$0 < \phi(x) \leqslant \omega(kx) \qquad (x \geqslant b), \tag{13·16}$$

for some fixed $k\,(1 \leqslant k \leqslant \infty)$. Then there is a sequence $x_1, x_2, \ldots$ tending to infinity, such that

$$\frac{\log \omega(x_n)}{\log x_n} \to \kappa, \tag{13·17}$$

and

$$\varliminf_{n \to \infty} \frac{\omega(x_n)}{\phi(x_n)} \geqslant k^{-\kappa}, \tag{13·18}$$

where

$$\kappa = \varlimsup_{x \to \infty} \frac{\log \omega(x)}{\log x}. \tag{13·19}$$

Suppose first that $k > 1$, $0 < \kappa < \infty$. It is sufficient to prove that, corresponding to each $\epsilon > 0$, there is a sequence $\xi_1, \xi_2, \ldots$ tending to infinity, such that

$$\omega(\xi_n) > \max\{(k^{-\kappa} - \epsilon)\,\phi(\xi_n), \xi_n^{\kappa-\epsilon}\}.$$

If this is false, then, for some value of ϵ less than $k^{-\kappa}$, and κ,

$$\omega(x) \leqslant \max\{(k^{-\kappa} - \epsilon)\,\phi(x), x^{\kappa-\epsilon}\} \qquad (x \geqslant c). \tag{13·20}$$

Take δ so that $0 < \delta < \epsilon$ and

$$l = (k^{-\kappa} - \epsilon)^{-1} > k^\eta \qquad \{\eta = \epsilon\,(\kappa - \delta)\,(\epsilon - \delta)^{-1}\}.$$

By (13·19) there is a sequence of points tending to infinity for which

$$\omega(x) \geqslant x^{\kappa - \delta}.$$

Let $X > \max(a, b, c)$ be a point of such a sequence, and let

$$Y = X^{(\kappa-\delta)/(\kappa-\epsilon)},$$

so that

$$(13\cdot21) \qquad \omega(x) \geqslant \omega(X) \geqslant X^{\kappa-\delta} = Y^{\kappa-\epsilon} \geqslant x^{\kappa-\epsilon} \qquad (X \leqslant x \leqslant Y).$$

Then (13·20), (13·21) give

$$l\omega(x) \leqslant \phi(x) \qquad (X \leqslant x \leqslant Y),$$

so that

$$l\omega(x) \leqslant \omega(kx) \qquad (X \leqslant x \leqslant Y),$$

and

$$(13\cdot22) \qquad l^n\omega(X) \leqslant \omega(k^n X) \qquad (k^{n-1}X \leqslant Y).$$

Let ν be the smallest integer such that

$$k^\nu > \frac{Y}{X} = X^{(\epsilon-\delta)/(\kappa-\epsilon)},$$

so that

$$\nu \log k > \frac{\epsilon-\delta}{\kappa-\epsilon} \log X \geqslant (\nu-1)\log k,$$

and

$$\nu \log k = \frac{\epsilon-\delta}{\kappa-\epsilon} \log X \left\{1 + O\left(\frac{1}{\nu}\right)\right\}.$$

Now, by (13·22),

$$\omega(k^\nu X) \geqslant l^\nu \omega(X) \geqslant l^\nu X^{\kappa-\delta},$$

so that

$$\begin{aligned}\log \omega(k^\nu X) &\geqslant \nu \log l + (\kappa-\delta)\log X \\ &= \left\{\frac{\epsilon-\delta}{\kappa-\epsilon}\frac{\log l}{\log k} + O\left(\frac{1}{\nu}\right) + \kappa - \delta\right\}\log X,\end{aligned}$$

while

$$\log(k^\nu X) = \left\{\frac{\epsilon-\delta}{\kappa-\epsilon} + O\left(\frac{1}{\nu}\right) + 1\right\}\log X.$$

On dividing these and making $X \to \infty$, so that $\nu \to \infty$, we get

$$\begin{aligned}\overline{\lim_{x\to\infty}} \frac{\log\omega(x)}{\log x} &\geqslant \left\{\frac{\epsilon-\delta}{\kappa-\epsilon}\frac{\log l}{\log k} + \kappa - \delta\right\}\left\{\frac{\epsilon-\delta}{\kappa-\epsilon} + 1\right\}^{-1} \\ &= \frac{\epsilon-\delta}{\kappa-\delta}\frac{\log l}{\log k} + \kappa - \epsilon > \frac{(\epsilon-\delta)}{(\kappa-\delta)}\frac{\epsilon(\kappa-\delta)}{(\epsilon-\delta)} + \kappa - \epsilon = \kappa,\end{aligned}$$

a contradiction.

If $k=1$ or $\kappa=\infty$ the result is trivial, and the case $\kappa=0$ is dealt with by consideration of the functions

$$\omega_1(x)=x^\lambda\omega(x), \quad \phi_1(x)=x^\lambda\phi(x),$$

where $\lambda>0$.

Let $f(z)$ be an integral function of order ρ and let

$$T(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta,$$

$$(13\cdot23) \quad S(a,r)=\frac{2}{(r^2-a^2)}\int_a^r T(u)\,u\,du.$$

It is known that*

$$\log M(r)\leqslant\frac{k+1}{k-1}T(kr) \qquad (k>1),$$

and that $T(r)$ is an increasing function of r, so that $S(a,r)$ is an increasing function of r for fixed a, and an increasing function of a for fixed r. Hence

$$\begin{aligned} S(a,kr) &\geqslant S(0,kr) \qquad (a>0,k>1)\\ &\geqslant\frac{2}{k^2r^2}\int_{\sqrt{kr}}^{kr}T(u)\,u\,du\geqslant\frac{2}{k^2r^2}T(\sqrt{kr})\int_{\sqrt{kr}}^{kr}u\,du\\ &\geqslant\frac{k-1}{k}\frac{\sqrt{k}-1}{\sqrt{k}+1}\log M(r).\end{aligned}$$

Use L_{509} and take $k=(1+1/\rho)^2$. Then follows:

L_{510}. *If $f(z)$ is an integral function of finite order ρ, there is a sequence $r_1, r_2, \ldots$ tending to infinity, such that*

$$(13\cdot24) \qquad \frac{\log S(a,r_n)}{\log r_n}\to\rho,$$

and

$$(13\cdot25) \qquad \lim_{n\to\infty}\frac{S(a,r_n)}{\log M(r_n)}\geqslant\frac{\rho^{2\rho}}{(\rho+1)^{2\rho+2}}.$$

We are now in a position to deal with certain functions of order 2 (**3**, **6**).

* Nevanlinna (**1**), 24.

Theorem 18. *If $f(z)$ is an integral function of order $\rho \leqslant 2$, satisfying the condition*

$$n(r) = o(r^2), \tag{13·26}$$

and $\eta\,(<\rho)$ and d are given positive numbers, the values of ζ for which the inequality

$$\log|f(z)| > \tfrac{1}{100}\log M(|\zeta|) > |\zeta|^{\eta} \qquad (|z-\zeta| \leqslant d) \tag{13·27}$$

is satisfied form a set of upper density greater than or equal to $\frac{1}{100}$.

Given $\epsilon > 0$, let $r_1, r_2, \ldots$ be a sequence for which (in accordance with L_{510})

$$S(r_n) = S(0, r_n) > (H-\epsilon)\log M(r_n) \qquad \left(H = \frac{2^4}{3^6} = \frac{16}{729}\right), \tag{13·28}$$

and

$$\log M(r_n) > \frac{100 r_n^{\eta}}{100h-1}, \tag{13·29}$$

where $\frac{1}{50} > h > \frac{1}{100}$. Now write $t\pi r_n^2 =$ measure of set of ζ in $|z| \leqslant r_n$ for which

$$\log|f(\zeta)| > h\log M(r_n), \tag{13·30}$$

so that

$$\pi r_n^2 S(r_n) \leqslant t\pi r_n^2 \log M(r_n) + (1-t)\pi r_n^2 h \log M(r_n).$$

Comparing this with (13·28), we see that

$$t + (1-t)h > H - \epsilon,$$

so that

$$t > \frac{H-h-\epsilon}{1-h}.$$

As ϵ is arbitrary, it follows from (13·29), (13·30) that the set of ζ for which

$$\log|f(\zeta)| > h\log M(|\zeta|) > \frac{100h}{100h-1}|\zeta|^{\eta} \tag{13·31}$$

has upper density greater than or equal to $(H-h)/(1-h)$.

There are now three cases to be considered, according as

$$1 < \rho \leqslant 2, \quad \rho = 1, \quad \text{or} \quad \rho < 1.$$

In the first case, η may be supposed greater than 1 and δ so chosen that $0 < \delta < \eta - 1$.

By L_{506}, (13·11), (13·31) are simultaneously true for a set of upper density greater than or equal to $(H-h)/(1-h)$, and for this set

$$\begin{aligned} \log|f(z)| &> h\log M(|\zeta|)-|\zeta|^{1+\delta} \qquad (|z-\zeta|\leqslant d)\\ &> h\log M(|\zeta|)-(h-\tfrac{1}{100})\log M(|\zeta|)\\ &= \tfrac{1}{100}\log M(|\zeta|), \end{aligned} \tag{13·32}$$

and also

$$\tfrac{1}{100}\log M(|\zeta|)>|\zeta|^{\eta}. \tag{13·33}$$

(13·32), (13·33) do not involve h. Hence the upper density of the set of ζ for which they are true is greater than or equal to

$$(H-\tfrac{1}{100})/(1-\tfrac{1}{100})>\tfrac{1}{100}.$$

The other cases can be treated in a similar manner with the aid of L_{508}.

L_{502} is an obvious deduction, as the hypothesis of Theorem 18 is implied by (12·6) (the converse is false).

Theorem 18 can be extended to functions of any finite order by a kind of change of variable process. If σ is a given positive number less than $1+\frac{1}{2}\rho$, the inequality

$$\log|f(z)|>K\log M(|\zeta|) \qquad (|z-\zeta|\leqslant|\zeta|^{\sigma})$$

is satisfied for arbitrarily large values of $|\zeta|$ (4).

CHAPTER VI

ASYMPTOTIC PERIODS

§ 14. Integral functions.

The results of the present chapter, in a sense converses of those of Chapter II, are concerned with the order of the difference,

$$(14\cdot 1) \qquad \Delta_\omega(z) = f(z+\omega) - f(z) \qquad (\omega \neq 0),$$

of an integral or meromorphic function.

$\Delta_\omega(z)$ may be identically zero. ω is then a period of $f(z)$, and it is well known that an integral function may either have no periods or else a single sequence $k\lambda\,(k = \pm 1, \pm 2, \ldots)$. The values of ω for which $\Delta_\omega(z)$ is "smaller" than $f(z)$ may however form sets of a more complicated kind, e.g. the integral function

$$R(z) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n!\, z}}{(n!)!}$$

is of infinite order, but, if p is any rational number,

$$R(z+p) - R(z)$$

is of order 1. Numbers with this property will be called *asymptotic periods*. More precisely:

Definition: *A number* $\beta\,(\neq 0)$ *is an asymptotic period of an integral or meromorphic function* $f(z)$ *if* $\Delta_\beta(z)$ *is of lower order than* $f(z)$.

This definition includes the case of a function $f(z)$ of infinite order. $\Delta_\beta(z)$ must then be of finite order. It is evident that, if β, γ are asymptotic periods, then $-\beta$, $\beta+\gamma$ have the same property. The main result for integral functions is as follows:

The asymptotic periods of an integral function form a linear set of measure zero.

Before embarking on the proof of this result we will construct an example showing that the set may be non-enumerable (**9**).

Let $\psi(n)$ be a positive increasing function such that, for every $\delta > 0$,

$$\sum_1^\infty e^{-\psi(n)+nr} < e^{r^{1+\delta}} \qquad (r \geqslant r_\delta),$$

e.g. $\psi(n) = e^n$; and let integer sequences $\{c_n\}$, $\{d_n\}$ be defined by the recurrence relations

$$c_1 = 2, \quad d_n = c_n!, \quad c_{n+1} = [2\psi(d_n)],$$

where $[x]$ denotes the greatest integer not exceeding x.

The set of numbers $\quad \pi\left\{\frac{b_1}{d_1} + \frac{b_2}{d_2} + \dots\right\},$

where $b_1, b_2, \dots$ are each either 0 or 1, will be denoted by E_ψ. E_ψ is a set of cardinal c. If x is a member of it,

$$|\sin d_n x| < e^{-\psi(d_n)} \qquad (n \geqslant 1),$$

since
$$|\sin d_n x| = \left|\sin \pi d_n \left\{\frac{b_{n+1}}{d_{n+1}} + \frac{b_{n+2}}{d_{n+2}} + \dots\right\}\right|$$
$$\leqslant \pi d_n \left\{\frac{b_{n+1}}{d_{n+1}} + \dots\right\} < 2\pi \frac{d_n}{d_{n+1}} < 2\pi c_n^{c_n - c_{n+1}} < e^{-\psi(d_n)}.$$

Given $\rho > 1$, the coefficients A_n $(0 < A_n \leqslant 1)$ can be chosen so that

$$F(z) = \sum_{n=1}^\infty A_n e^{d_n iz}$$

is an integral function of order ρ. If α is a member of E_ψ,

$$|F(z+2\alpha) - F(z)| = \left|\sum_1^\infty (e^{2d_n i\alpha} - 1) A_n e^{d_n iz}\right|$$
$$\leqslant 2\sum_1^\infty |\sin d_n \alpha| A_n e^{d_n r} \qquad (r = |z|)$$
$$< 2\sum_1^\infty e^{-\psi(d_n)+d_n r} < 2\sum_1^\infty e^{-\psi(n)+nr}$$
$$< 2e^{r^{1+\delta}} \qquad (r \geqslant r_\delta),$$

so that $\Delta_{2\alpha}(z)$ is of order 1.

A function of any order greater than 1 may therefore have a non-enumerable set of asymptotic periods. It will appear that a function of order less than 1 cannot have any, and a function of order 1 either none or else a sequence $k\lambda$. The results may be summed up as follows (**7**):

Theorem 19. *If B denotes the set of asymptotic periods of an integral function $f(z)$, either*

(i) *B is null; or*

(ii) *B consists of a set of points $k\lambda$, $k = \pm 1, \pm 2, \ldots$; or*

(iii) *B lies on a straight line through the origin, is everywhere dense, and has measure zero.*

If

$$\frac{\log M(r)}{r} \to 0, \tag{14·2}$$

only case (i) *is possible; and, if $f(z)$ is of order* 1, *only cases* (i) *and* (ii).

Proof.

$\mathbf{L}_{601}$. *The ratio of two asymptotic periods of an integral function is real.*

If possible, let β, γ be asymptotic periods whose ratio is not real. It follows from the identity

$$f(z+k\beta+l\gamma)=f(z)+\sum_{s=0}^{l-1}\Delta_\gamma(z+s\gamma)+\sum_{t=0}^{k-1}\Delta_\beta(z+l\gamma+t\beta) \tag{14·3}$$

that, if $|z|=r$,

$$f(z)=f(z_0)+\sum_s \Delta_\gamma(z_s)+\sum_t \Delta_\beta(z_t'),$$

where z_0 is in a "period parallelogram" with the origin as one vertex, $z_1, z_2, \ldots$ are points inside the circle $|z|=r$, each in a different "period parallelogram", and the same is true of $z_1', z_2', \ldots$. In each case the number of points does not exceed $K(\beta,\gamma)r$. Hence

$$M(r) \leqslant M(|\beta|+|\gamma|)+K(\beta,\gamma)r\{M_\beta(r)+M_\gamma(r)\},$$

where $M_\beta(r)$, $M_\gamma(r)$ denote the maximum moduli of $\Delta_\beta(z)$, $\Delta_\gamma(z)$, and this is evidently incompatible with the hypothesis that $\Delta_\beta(z)$, $\Delta_\gamma(z)$ are of lower order than $f(z)$.

$\mathbf{L}_{602}$. *Let $f(z)$ be an integral function of period k and let*

$$\varliminf_{r\to\infty}\frac{\log M(r)}{r} < \frac{2\pi}{|k|}. \tag{14·4}$$

Then $f(z)$ is a constant.

There is no loss of generality in taking $k=2\pi$. Consider the function $f(-i\log w)$. This is regular and uniform in every annulus $0<R\leqslant |w|\leqslant R'$, and so can be expanded in a Laurent series

$$\sum_{n=0}^{\infty} a_n w^n + \sum_{n=1}^{\infty} b_n w^{-n},$$

so that, writing $z=-i\log w$,

$$f(z)=\sum_{n=0}^{\infty} a_n e^{niz}+\sum_{n=1}^{\infty} b_n e^{-niz}$$

$$=F_1(e^{iz})+F_2(e^{-iz}),$$

where $F_1(u)$, $F_2(u)$ are integral functions. Write

$$M_1(r)=\max_{|u|=r}|F_1(u)|,\quad M_2(r)=\max_{|u|=r}|F_2(u)|.$$

Mark the points

$$A(\pi+iy),\quad B(-\pi+iy),\quad C(-\pi-iy),\quad D(\pi-iy)$$

in the z-plane and draw the circle circumscribing $ABCD$, i.e. the circle of centre the origin, radius $\sqrt{(\pi^2+y^2)}$. Now

$$\max_{CD}|F_1(e^{iz})|=M_1(e^y),\quad \max_{CD}|F_2(e^{-iz})|\leqslant\sum_1^{\infty}|b_n|e^{-ny}\leqslant\sum_1^{\infty}|b_n|,$$

so that, denoting by $M(r)$ the maximum modulus of $f(z)$,

$$M_1(e^y)\leqslant M(\sqrt{\pi^2+y^2})+\sum_1^{\infty}|b_n|,$$

and so, by (14·4),

$$\varlimsup_{y\to\infty}\frac{\log M_1(e^y)}{y}<1,$$

i.e.

$$\varlimsup_{r\to\infty}\frac{\log M_1(r)}{\log r}<1. \tag{14·5}$$

This is only possible if $F_1(u)$ is a constant, the expression on the left of (14·5) being infinite if $F_1(u)$ is transcendental, and N if $F_1(u)$ is a polynomial of degree N. Similarly $F_2(u)$ is a constant.

We can now deal with functions of order 1 or less. Suppose first that (14·2) is satisfied. If there is an asymptotic period β,

$\Delta_\beta(z)$ must be of order $\sigma < 1$, and so, by Theorem 3, there is a function $f_1(z)$ of order σ such that

$$\Delta_\beta(z) = f_1(z+\beta) - f_1(z).$$

$f_2(z) = f(z) - f_1(z)$ is therefore an integral function of period β and

$$\overline{\lim_{r\to\infty}} \frac{\log M_2(r)}{r} \leqslant \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r} + \overline{\lim_{r\to\infty}} \frac{\log M_1(r)}{r} = 0.$$

L_{602} shows that $f_2(z)$ is a constant, and this is impossible since $f_1(z)$ is of lower order than $f(z)$. The contradiction implies that $f(z)$ has no asymptotic periods.

Next suppose that $f(z)$ is of order 1, but not of minimum type. If there are asymptotic periods one of them may be taken to be 2π. As before,

$$f(z) = f_1(z) + f_2(z),$$

where $f_2(z)$ is of order 1 and has period 2π, and $f_1(z)$ is of order $\sigma < 1$; and, as in L_{602},

$$f_2(z) = a_0 + \sum_1^\infty (a_n e^{niz} + b_n e^{-niz}).$$

$f_2(z)$ is not a constant, so not all the terms of the series are identically zero. Let N be the rank of the first that is not. Now, if β is any other asymptotic period,

$$\sum_1^\infty (a_n e^{niz} - b_n e^{-niz-ni\beta})(e^{ni\beta} - 1) = f_2(z+\beta) - f_2(z)$$
$$= \text{function of order less than } 1.$$

This can only be so if every term in the series is identically zero, and so

$$e^{Ni\beta} - 1 = 0.$$

Thus $\beta = 2\pi k/N = \lambda k$, where k is an integer.

L_{603}. *Let E be a non-null linear set of points with the property that if x, y are any members of E, $-x$, $x+y$ are also members. Then either (a) E consists of a set of points $k\lambda$, or (b) E is everywhere dense.*

If E is not everywhere dense, let (α, β) be an interval not containing a member of E. Let α' be the upper bound of members of E not greater than α, and β' the lower bound of members of E not less than β. Then there are no members of E inside the interval $(\beta', 2\beta'-\alpha')$. For suppose that such a number, x, exists. As β' is a lower bound of members of E there is a member y satisfying the inequality

$$\beta' \leqslant y < \tfrac{1}{2}(x+\beta').$$

By the addition property $2y-x$ is a member of E, and

$$\alpha' < 2\beta' - x \leqslant 2y - x < \beta',$$

giving a contradiction. Similarly there are no members of E inside any interval

$$\{\beta' + k(\beta'-\alpha'),\ \beta' + (k+1)(\beta'-\alpha')\},$$

and so E consists of the points $\beta' + k(\beta'-\alpha')$, $k=0, \pm 1, \ldots$. As the origin is a member of E it is evident that this set is of the form $k\lambda$, $k=0, \pm 1, \pm 2, \ldots$.

The next lemma contains the kernel of the argument. Let $F(x,y)$ be a complex function of two real variables, defined in $(-\infty < x < \infty,\ y \geqslant a)$. Let its *order* be defined to be

$$\rho = \varlimsup_{y\to\infty} \frac{\log\log\psi(y)}{\log y} \qquad \left(\psi(y) = \max_{-\infty<x<\infty} |F(x,y)|\right),$$

and let γ be called *exceptional* if $F(x+\gamma,y) - F(x,y)$ is of order less than ρ.

L$_{604}$. *If $F(x,y)$, of order ρ, is continuous and periodic with respect to x and if*

$$(14\cdot6)\quad \varlimsup_{y\to\infty} \frac{\log\log\phi(y)}{\log y} = \rho \quad \left(\phi(y) = \max_{-\infty<x,\,x'<\infty} |F(x,y) - F(x',y)|\right)$$

the exceptional numbers form a set of measure zero.

Let B_μ $(\mu < \rho)$ denote the set of numbers γ such that

$$F(x+\gamma,y) - F(x,y)$$

is of order less than or equal to μ. The sets B_μ form an increasing sequence whose limit is B, the set of all exceptional numbers.

Let N be a "large" positive integer and let $\mu < \beta < \alpha < \rho$. There is a sequence $y_1, y_2, \ldots$ tending to ∞, such that

$$(14\cdot7)\qquad \phi(y_p) > \exp y_p{}^\alpha \qquad (p=1,2,\ldots),$$

and also

$$(14\cdot8)\qquad y_p{}^\alpha > y_p{}^\beta + N + p + 1.$$

Corresponding to y_p, a member of this sequence, let x_1, x_2 be points of $(0, l)$, where l is the period of $F(x, y)$, for which

$$|F(x_1, y_p) - F(x_2, y_p)| = \phi(y_p).$$

Mark in the F-plane the values of $F(x, y)$ corresponding to values of (x, y_p), $0 \leqslant x \leqslant l$. The resulting curve is continuous and closed, since $F(x, y)$ has period l, and is contained in the circle of centre $F(x, y_p)$, radius $\phi(y_p)$. Now divide this circle into annuli by drawing concentric circles of radii $s\phi(y_p)/S$, $(s = 1, 2, \ldots, S-1)$, where $S = 2^{N+p}$, and denote by E_s the set of points x for which the F-curve lies in the sth annulus. To make matters precise, let E_s consist of the points x in $(0, l)$ for which

$$(s-1)S^{-1}\phi(y_p) \leqslant |F(x_1, y_p) - F(x, y_p)| < sS^{-1}\phi(y_p) \qquad (s = 1, 2, \ldots, S-1),$$
$$\leqslant \phi(y_p) \qquad (s = S).$$

Evidently $$mE_1 + mE_2 + \ldots + mE_S = l,$$
so that there is at least one value of k for which

$$mE_k \leqslant lS^{-1}.$$

As the F-curve is continuous and passes from the centre of the circle of radius $\phi(y_p)$ to its circumference, it must cut the concentric circle of radius $(k - \frac{1}{2})S^{-1}\phi(y_p)$. That is to say, there is a point ξ_p in $(0, l)$ such that

$$|F(\xi_p, y_p) - F(x_1, y_p)| = (k - \tfrac{1}{2})S^{-1}\phi(y_p).$$

On drawing a diagram it is evident that, if x is a point of E_s, $(s \neq k)$, then $$|F(\xi_p, y_p) - F(x, y_p)| \geqslant \tfrac{1}{2}S^{-1}\phi(y_p),$$
and hence the values of x in $(0, l)$ for which this inequality is satisfied form a set $E^{(p)}$ of measure not less than $(1 - S^{-1})l$. As $F(x, y)$ has period l this means that the values of t in $(0, l)$ for which $$|F(\xi_p, y_p) - F(\xi_p + t, y_p)| \geqslant \tfrac{1}{2}S^{-1}\phi(y_p)$$
form a set T_p of measure not less than $(1 - S^{-1})l$. Let

$$T = T_1 T_2 T_3 \ldots,$$

so that

$$mT \geqslant \left(1 - \sum_{p=1}^{\infty} 2^{-p-N}\right) l = (1 - 2^{-N})l. \tag{14·9}$$

It will now be shown that the sets T, B_μ have no members in common.

If TB_μ is not null, let γ be a member of it. Then, for all values of p,

$$\begin{aligned}\psi_0(y_p) &= \max_{-\infty<x<\infty} |F(x+\gamma, y_p) - F(x, y_p)| \\ &\geqslant |F(\xi_p+\gamma, y_p) - F(\xi_p, y_p)| \\ &\geqslant \tfrac{1}{2}S^{-1}\phi(y_p) > \tfrac{1}{2}S^{-1}\exp y_p{}^\alpha > \exp y_p{}^\beta,\end{aligned}$$

by (14·8), and this is false, since it implies that

$$\overline{\lim_{y\to\infty}} \frac{\log\log\psi_0(y)}{\log y} \geqslant \beta > \mu,$$

whereas $F(x+\gamma, y) - F(x, y)$ is of order less than or equal to μ. Hence the set TB_μ is null, and so if $B_\mu^\star$ denotes the part of B_μ in the interval $(0, l)$, it follows by (14·9) that

$$mB_\mu^\star \leqslant 2^{-N} l.$$

As N is independent of l, μ, this implies that

$$mB_\mu^\star = 0,$$

and so
$$mB_\mu = 0$$

for each μ. Hence, finally,

$$mB = 0.$$

The condition (14·6) is essential, e.g. every number is exceptional with respect to the function

$$F(x, y) = e^{y^\lambda} + e^{y^\mu}\sin x \qquad (\mu < \lambda),$$

but of course we have assumed very little about $F(x, y)$, not even continuity with respect to y. The remaining lemma enables us to dispense with (14·6) in the case of an integral function.

L$_{605}$. *Let $f_1(z)$, $f_2(z)$ be periodic integral functions of order ρ, whose periods ω, ω/μ are real, and let*

$$g(z) = f_1(z) - f_2(z).$$

If, for some $\sigma < \rho$,

$$(14\cdot10) \qquad |g(x+iy)| < Ke^{|y|^\sigma} \qquad (|y| \geqslant a),$$

where K is independent of x, then μ is a rational number.

ρ may be ∞. As in L_{602}, take $\omega = 2\pi$ so that

$$(14\cdot11)\qquad f_1(z) = \sum_{n=0}^{\infty} a_n e^{niz} + \sum_{n=1}^{\infty} b_n e^{-niz},$$

$$(14\cdot12)\qquad f_2(z) = \sum_{n=0}^{\infty} c_n e^{n\mu iz} + \sum_{n=1}^{\infty} d_n e^{-n\mu iz}.$$

As $f_1(z)$ is of order ρ at least one of the series in (14·11), say the first, is an integral function of order ρ.

Now, if μ is irrational, the exponentials in (14·11), (14·12) are all distinct, and so

$$a_n e^{niz} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{ni(x-t)} g(t+iy)\, dt,$$

and thus, if $y \geqslant 0$,

$$(14\cdot13)\qquad |a_n| e^{ny} \leqslant \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |g(t+iy)|\, dt \leqslant G(y),$$

where
$$G(y) = \max_{-\infty<x<\infty} |g(x \pm iy)|.$$

By (14·10),

$$(14\cdot14)\qquad G(y) < Ke^{y^\sigma} \qquad (y \geqslant a).$$

Now the function

$$F(w) = \sum_{n=0}^{\infty} a_n w^n \qquad (w = Re^{i\phi})$$

is an integral function. Let $A(R)$ denote its maximum modulus,

$$A(R) = \max_{|w|=R} |F(w)|,$$

γ its order, and

$$\alpha(R) = \max_{n\geqslant0} |a_n| R^n$$

its maximum term. (14·13) shows that

$$(14\cdot15)\qquad \alpha(e^y) \leqslant G(y) \qquad (y \geqslant 0).$$

Moreover, if $M(r)$ denotes the maximum modulus of $F(e^{iz})$ for $|z| = r$,

$$(14\cdot16)\qquad M(y) \leqslant A(e^y) \qquad (y \geqslant 0).$$

For $M(y)$ is the maximum of $|F(w)|$ on the curve

$$R = e^{-y\sin\theta}, \quad \phi = y\cos\theta \qquad (0 \leqslant \theta \leqslant 2\pi)$$

in the w-plane, and this lies inside the curve

$$R = e^y.$$

Now, if γ is finite,*

$$(14\cdot17) \qquad \frac{\log \alpha(r)}{\log A(r)} \to 1 \qquad \text{as } r \to \infty,$$

and so, by (14·15), (14·16),

$$\varliminf_{y \to \infty} \frac{\log G(y)}{\log M(y)} \geqslant 1.$$

Moreover
$$\varlimsup_{y \to \infty} \frac{\log \log M(y)}{\log y} = \rho.$$

Hence
$$\varlimsup_{y \to \infty} \frac{\log \log G(y)}{\log y} \geqslant \rho > \sigma,$$

and this contradicts (14·14).

Again, if γ is infinite,†

$$(14\cdot18) \qquad \varlimsup_{r \to \infty} \frac{\log \log \alpha(R)}{\log R} = \infty,$$

whence, by (14·15),

$$\varlimsup_{y \to \infty} \frac{\log \log G(y)}{y} = \infty,$$

which likewise contradicts (14·14).

The proof of the main theorem can now be completed. Let β be an asymptotic period, supposed real. As in L_{602},

$$f(z) = f_1(z) + f_2(z),$$

where $f_2(z)$ has period β and is of order ρ (possibly ∞) and $f_1(z)$ is of order σ less than ρ. Write

$$\phi(y) = \max_{0 \leqslant x,\, x' \leqslant \beta} |f_2(x+iy) - f_2(x'+iy)|.$$

It may be that

$$(14\cdot19) \qquad \phi(y) \leqslant e^{|y|^\tau} \qquad (|y| \geqslant a)$$

for some $\tau < \rho$. In this case every other asymptotic period is a rational multiple of β. For, if γ is any other,

$$f_2(z) = f_3(z) + g(z),$$

* Valiron (**1**), 32.

† Cf. Valiron (**1**). (14·18) is not explicitly stated but follows readily from equation (2·9) on p. 31.

where $f_3(z)$ has period γ and is of order ρ and $g(z)$ is of order $\sigma' < \rho$. We may suppose that τ in (14·19) is greater than σ, σ'. Then, when $|y| \geqslant b$,

$$(14{\cdot}20) \qquad \phi^\star(y) = \max_{0 \leqslant x, x' \leqslant \gamma} |f_3(x+iy) - f_3(x'+iy)| \leqslant 2e^{|y|^\tau}.$$

For, if this is false, then corresponding to a given value of K, numbers x, x', y can be found such that

$$0 \leqslant x,\ x' \leqslant \max(\beta, \gamma), \qquad |y| > K,$$

and

$$|g(x+iy) - g(x'+iy)|$$
$$\geqslant |f_3(x+iy) - f_3(x'+iy)| - |f_2(x+iy) - f_2(x'+iy)| > e^{|y|^\tau}.$$

Hence, writing $\quad M_0(r) = \max_{|z|=r} |g(z)|,$

$$2M_0(2|y|) \geqslant |g(x+iy)| + |g(x'+iy)| > e^{|y|^\tau},$$

and this is impossible if K is sufficiently large, as the order of $g(z)$ does not exceed σ'.

(14·20) being thus established it follows that, if x, x' are any real numbers, then

$$\begin{aligned} |g(x+iy) - g(x'+iy)| &\leqslant |f_2(x+iy) - f_2(x'+iy)| \\ &\qquad + |f_3(x+iy) - f_3(x'+iy)| \\ &\leqslant \phi(y) + \phi^\star(y) \leqslant 3e^{|y|^\tau} \qquad (|y| \geqslant c), \end{aligned}$$

and so, if $-\infty < x < \infty$,

$$\begin{aligned} |g(x+iy)| &\leqslant |g(iy)| + 3e^{|y|^\tau} \\ &< 4e^{|y|^\tau} \qquad (|y| \geqslant d), \end{aligned}$$

since $g(z)$ is of order $\sigma' < \tau$.

It now follows from L_{605} that γ is a rational multiple of β.

If (14·19) is false for every $\tau < \rho$, (14·6) is satisfied either for $F(x, y) = f_2(x+iy)$ or for $F(x, y) = f_2(x-iy)$, say for the former. Let γ be another asymptotic period. Then, if $0 \leqslant x \leqslant \beta$,

$$\begin{aligned} |F(x+\gamma, y) - F(x, y)| &= |f_2(x+\gamma+iy) - f_2(x+iy)| \\ &= |g(x+\gamma+iy) - g(x+iy)| \\ &< 2M_0(2y) < \exp y^{\sigma'+\epsilon}, \end{aligned}$$

and this is true for all x, as $F(x+\beta, y) = F(x, y)$. Hence γ is an exceptional number, and we can apply L_{604}.

§ 15. Meromorphic functions.

The main result for meromorphic functions is as follows (**10**):

Theorem 20. *Let B denote the set of asymptotic periods of a meromorphic function of order ρ, and let κ denote the exponent of convergence of the poles, so that $\kappa \leqslant \rho$. Then*

(*a*) *if $\kappa = \rho$, B is an enumerable set;*

(*b*) *if $\kappa < \rho$, B lies on a straight line through the origin and has measure zero.*

The proof of (*a*) is immediate.

The order of $\Delta_\beta(z)$ is not inferior to the exponent of convergence of its poles. Now these consist of the poles $b_1, b_2, \ldots$ of $f(z)$ together with the poles $b_1-\beta$, $b_2-\beta$, ... of $f(z+\beta)$, and so have exponent of convergence ρ, unless some of the poles cancel out. Thus β cannot be an asymptotic period unless it is of the form $b_p - b_q$. It follows that B must be enumerable. It should be noted that not every number of the form $b_p - b_q$ need be an asymptotic period.

As regards descriptive properties it can be shown by the methods of L_{603} that there are six possibilities:

(α) B is null;

(β) B consists of a set of points $k\omega$, $k = \pm 1, \pm 2, \ldots$;

(γ) B lies on a straight line through the origin and is everywhere dense on it;

(δ) B consists of a set of points $k\omega + l\omega'$, where ω'/ω is not real;

(ϵ) B lies on a set of lines $y = mx + kc$, $k = 0, \pm 1, \pm 2, \ldots$, and is everywhere dense on each of them;

(ζ) B is everywhere dense in the plane.

All these possibilities are realised among functions of any positive order, e.g. if $c = 2^{1/\rho}$ $(0 < \rho < 1)$ and

$$\phi(z) = \sum_{n=1}^{\infty} \prod_{k,l=1}^{4^n} \left(1 - \frac{z}{c^{4n} + \dfrac{k}{2^n} + \dfrac{li}{2^n}}\right), \tag{15·1}$$

$\phi'(z)/\phi(z)$ is a meromorphic function of order ρ for which every number of the form $2^{-n}(k+li)$, where k, l, n are integers, is an asymptotic period.

The proof of (*b*) is much more difficult.

L_{606}. *Let $f(z)$ be an integral function of finite order ρ. Then corresponding to each number $\sigma>\rho$ there is a number d_σ such that*

$$(15\cdot 2)\quad \iint_A \log^+|f(z)|^{-1}\,dS \leqslant \sum_A r_n^{-\sigma} + D^\sigma S(A) \qquad (d\geqslant d_\sigma),$$

where A is any domain in the complex plane, $S(A)$ its area, $r_1, r_2, \ldots$ the moduli of the zeros of $f(z)$, the summation is taken over the zeros inside or on the boundary of A (multiple zeros being counted multiply), and

$$d=\min|z|,\quad D=\max|z| \qquad (z \text{ in } A).$$

The function $f(z)$ is of the form

$$P(z)\exp(k_1z+k_2z^2+\ldots+k_sz^s),$$

where $P(z)$ is a canonical product and $s\leqslant\rho$; so that, writing $|z|=r$,

$$\log^+|f(z)|^{-1}\leqslant\log^+|P(z)|^{-1}+|k_1|r+\ldots+|k_s|r^s.$$

Now

$$(15\cdot 3)\quad \iint_A (|k_1|r+\ldots+|k_s|r^s)\,dS<\tfrac{1}{2}D^\sigma S(A) \qquad (d\geqslant d_\sigma').$$

Moreover*

$$(15\cdot 4)\quad \log|P(z)|>-KI+\log\prod_1^N\left|1-\frac{z}{a_n}\right| \qquad (r_N\leqslant kr<r_{N+1}),$$

where $k>1$, and

$$(15\cdot 5)\qquad I=\int_0^\infty \frac{n(x)}{x^{p+1}}\,\frac{r^{p+1}}{x+r}\,dx<r^{\rho+\epsilon} \qquad (r\geqslant r_\epsilon),$$

p being the genus of $P(z)$. (15·4), (15·5) give

$$\log^+|P(z)|^{-1}<KI+\sum_1^N\log^+r_n+\sum_1^N\log^+|z-a_n|^{-1}$$

$$<\tfrac{1}{2}D^\sigma+\sum_1^N\log^+|z-a_n|^{-1} \qquad (r\geqslant r_\sigma),$$

and so

$$(15\cdot 6)\quad \iint_A\log^+|P(z)|^{-1}dS<\tfrac{1}{2}D^\sigma S(A)+\sum_1^N\iint_A\log^+|z-a_n|^{-1}dS.$$

* Valiron (**1**), 53.

Now surround each zero a_n with a circle C_n of radius r_n^{-h} $(h > \frac{1}{2}\sigma)$, and let A_n be the domain A with C_n excluded. Then

$$\iint_{A_n} \log^+ |z - a_n|^{-1} dS \leqslant h \log r_n \,.\, S(A),$$

and, if $r_n > 1$,

$$\iint_{C_n} \log^+ |z - a_n|^{-1} dS = \int_0^{r_n^{-h}} \int_0^{2\pi} \log u^{-1}\, u\, du\, d\theta$$

$$= \pi (h \log r_n + \tfrac{1}{2}) r_n^{-2h} < r_n^{-\sigma} \qquad (d \geqslant d_\sigma'').$$

On combining these inequalities with (15·3) and (15·6) we get the result stated.

Now take the case (*b*). $f(z)$ is of the form

$$f(z) = \frac{h(z)}{g(z)},$$

where $g(z)$ is a canonical product of order $\kappa < \rho$ and $h(z)$ is an integral function of order ρ. If β is an asymptotic period,

$$\Delta_\beta(z) = \frac{h(z+\beta)\, g(z) - g(z+\beta)\, h(z)}{g(z)\, g(z+\beta)}$$

is of lower order than $f(z)$, and so

$$F(z) = h(z+\beta)\, g(z) - g(z+\beta)\, h(z)$$

is of order $\lambda < \rho$.

Suppose, if possible, that γ is an asymptotic period which does not lie on the line joining 0, β. Then

$$G(z) = h(z+\gamma)\, g(z) - g(z+\gamma)\, h(z)$$

is of order $\mu < \rho$.

Let σ be chosen so that

$$(15\cdot7) \qquad \kappa, \lambda, \mu < \sigma < \rho,$$

and let d_σ be the associated number when L_{606} is applied to $g(z)$. Moreover, let $D_0 \geqslant d_\sigma$ be chosen so that

$$|g(z)|, \quad |F(z)|, \quad |G(z)| < e^{r^\sigma} \qquad (r \geqslant D_0).$$

The identity (14·3) shows that, if $D > D_0$ and z is a point in the annulus Γ ($D_0 \leqslant |z| \leqslant D$), then

$$f(z) = f(z_0) + \sum_s G(z_s)\{g(z_s)\,g(z_s+\gamma)\}^{-1} \\ + \sum F(z_t')\{g(z_t')\,g(z_t'+\beta)\}^{-1},$$

where z_0 is in a "period parallelogram" or cell (formed with periods β, γ) intersected by the circle $|z| = D_0$ and $z_1, z_2, \ldots$, $z_1', z_2', \ldots$ are points "conjugate" to z in different cells in the annulus. The number of these points does not exceed $K_{\beta,\gamma} D$, where $K_{\beta,\gamma}$ depends only on β, γ. Hence

$$|f(z)| \leqslant |f(z_0)| + K_{\beta,\gamma} D e^{D^\sigma} \max |g(z_v'')|^{-2},$$

where z_1'', z_2'', ... are all the points "conjugate" to z in the annulus.

On making use of the inequalities*

$$\log^+(\alpha_1\alpha_2\ldots\alpha_q) \leqslant \sum_1^q \log^+\alpha_s,$$

$$\log^+\left(\sum_1^q \alpha_s\right) \leqslant \sum_1^q \log^+\alpha_s + \log q,$$

this gives

$$(15\cdot8) \qquad \log^+|h(z)| \leqslant \log^+|g(z)| + \log^+|f(z)| \\ \leqslant \log^+|g(z)| + \log^+|h(z_0)| + \log^+|g(z_0)|^{-1} \\ + \log K_{\beta,\gamma} + \log D + D^\sigma \\ + 2\log^+ \max|g(z_v'')|^{-1} + \log 2.$$

Let Z denote the point (or one of the points) z_v'' for which $|g(z_v'')|^{-1}$ is a maximum. As z moves about in a cell, Z will jump from one cell to another, and as z moves over a whole cell, Z will move over a number of disconnected domains $A_1, A_2, \ldots, A_k$ in different cells. These can be fitted together, like pieces of a jig-saw puzzle, to make a whole cell, so

$$S(A_1) + S(A_2) + \ldots + S(A_k) = \text{area of cell} = S.$$

A single A_s may consist of several disconnected parts in the same cell.

* Nevanlinna (1), 14.

Now $$\log^+ | h(z_0) | \leqslant D_0^{\rho+\epsilon} < \tfrac{1}{2}D^\sigma \qquad (D \geqslant D_1),$$
so that (15·7), (15·8) give
$$\log^+ | h(z) | \leqslant 3 \log^+ | g(Z) |^{-1} + 2D^\sigma \qquad (D \geqslant D_1),$$
and, if C is any cell in the annulus Γ,
$$\iint_C \log^+ | h(z) | \, dS \leqslant 3 \sum_{s=1}^{k} \iint_{A_s} \log^+ | g(Z) |^{-1} dS + 2D^\sigma S.$$
On applying L_{606}, this gives (writing $| b_n | = r_n$)
$$\iint_C \log^+ | h(z) | \, dS \leqslant 3 \sum_{s=1}^{k} \left\{ \sum_{A_s} r_n^{-\sigma} + D^\sigma S(A_s) \right\} + 2D^\sigma S$$
$$\leqslant 12 \sum_{n=1}^{\infty} r_n^{-\sigma} + 5SD^\sigma < KD^\sigma,$$
the factor 12 being arrived at by observing that a pole b_n may be on the boundaries of four different domains A_s, if it happens to be the corner of a cell. Hence
$$\iint_\Gamma \log^+ | h(z) | \, dS < KD^\sigma \pi (D^2 - D_0^2) \qquad (D \geqslant D_1),$$
and so
$$\overline{\lim_{D\to\infty}} \, (\log D)^{-1} \log \frac{1}{\pi (D^2 - D_0^2)} \iint_\Gamma \log^+ | h(z) | \, dS \leqslant \sigma < \rho,$$
and this contradicts L_{510}. Hence the assumption that γ is not collinear with 0 and β is false, and so either B is null or B lies on a straight line through the origin.

We may therefore suppose that all the asymptotic periods are real, and it remains to prove that they form a set of measure zero.

By subtracting a suitable rational function, if necessary, the poles in $| z | \leqslant 3$ may be removed. The number of the remaining poles in $| z | \leqslant r$ satisfies an inequality
$$p(r) \leqslant r^A \qquad (A \geqslant 1) \text{ (all } r).$$
Surround each pole p_n with a circle C_n of radius $| p_n |^{-h}$, where h will be chosen later, and move the circles a distance $t \, (0 < t \leqslant 1)$ to the left, parallel to the real axis. Let C_n become $C_n'(t)$. It is evident from a diagram that the numbers t for which $C_1'(t)$ has a point in common with C_n form a set of measure less than or

equal to $2|p_1|^{-h}+2|p_n|^{-h}$, so $C_1'(t)$ will have a point in common with one or more of $C_1, C_2, \ldots$ for numbers t forming a set of measure

$$\mu_1 \leqslant \sum_{n=1}^{\nu_1}\{2|p_1|^{-h}+2|p_n|^{-h}\},$$

where ν_1 is the last integer such that

$$|p_{\nu_1}|-|p_{\nu_1}|^{-h}-|p_1|-|p_1|^{-h}\leqslant 1.$$

Thus $$|p_{\nu_1}|\leqslant 3+|p_1|,$$

and $$\nu_1 \leqslant (3+|p_1|)^A,$$

so that $$\mu_1 \leqslant \sum_{n=1}^{\nu_1}\{4|p_1|^{-h}\}\leqslant 4(3+|p_1|)^A|p_1|^{-h}.$$

Writing $h=A(D+1)$, where $D\geqslant 1$, this gives

$$\mu_1 \leqslant 4\{3|p_1|^{-D-1}+|p_1|^{-D}\}^A \leqslant 4\{2|p_1|^{-D}\}^A \leqslant 4.2|p_1|^{-D}.$$

$C_2'(t)$, $C_3'(t)$, ... can be discussed similarly and it follows that the numbers t in $0<t\leqslant 1$ for which some $C_n'(t)$ has a point in common with some C_m form a set E of measure not exceeding $8\Sigma|p_n|^{-D}$. By choice of D this can be made smaller than any given positive number δ.

With this choice of D, and $h=A(D+1)$, express $f(z)$ in normal form

$$f(z)=g(z)+\Sigma\left\{\frac{P(z)}{(z-b_0)^{\lambda_0}\ldots(z-b_k)^{\lambda_k}}+Q(z)\right\}.$$

We are supposing that $\kappa<\rho$, so $\rho=\max(\sigma,\tau_1)$. There are two cases to consider:

(i) $\tau_1<\rho$. Then $\sigma=\rho$ and $f(z)=g(z)+$ a function of order less than ρ. The asymptotic periods of $f(z)$ are therefore identical with those of $g(z)$ and it follows from Theorem 19 that the latter form a set of measure zero.

(ii) $\tau_1=\rho$. Unless t belongs to E,

$$\begin{aligned}f(z+t)-f(z)&=g(z+t)-g(z)\\&+\Sigma\left\{\frac{P(z+t)}{(z+t-b_0)^{\lambda_0}\ldots(z+t-b_k)^{\lambda_k}}+Q(z+t)\right\}\\&-\Sigma\left\{\frac{P(z)}{(z-b_0)^{\lambda_0}\ldots(z-b_k)^{\lambda_k}}+Q(z)\right\}\end{aligned}$$

will be in normal form, and so of order not less than $\tau_1 = \rho$. The asymptotic periods in $0 < t \leqslant 1$ are therefore members of E and so form a set of measure not exceeding δ. As δ is arbitrary, they form a set of measure zero.

The analogy between integral functions and meromorphic functions for which $\kappa < \rho$ suggests that, if $0 \leqslant \kappa < \rho < 1$, there should be no asymptotic periods, and, if $0 \leqslant \kappa < \rho = 1$, either none or else a single sequence $k\omega$. This is, in fact, the case.

The results of this chapter throw a certain amount of light on the nature of periodicity. The single periodicity of $\sin z$ and the double periodicity of $\operatorname{sn} z$ appear at first sight to be phenomena of much the same kind; but if we enlarge the concept by including asymptotic periodicity it becomes evident that they belong to classes of phenomena which differ in almost every respect. The asymptotic periodicity of an integral function, or a meromorphic function for which $\kappa < \rho$, is a Diophantine phenomenon limited to functions of order one or more; the asymptotic periods may be non-enumerable but are restricted to a line. In the case of a meromorphic function for which $\kappa = \rho$, on the other hand, the phenomenon is essentially connected with the poles and occurs among functions of any positive order; the asymptotic periods form an enumerable set but may be everywhere dense in the plane.

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